WORKS OF

FREDERIC H. MILLER

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PARTIAL DIFFERENTIAL EQUATIONS

BY

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PREFACE

The subject of partial differential equations, approached from an elementary standpoint, has been much neglected in modern mathematical textbooks. Previously published books, used for first courses in differential equations, usually either ignore partial equations or devote a chapter or two to a brief and inadequate treatment of this branch of mathematics. This is regrettable, for to the student engaged with any one of three broad fields—analysis, geometry, or the physical sciences—partial differential equations have an importance comparable to that of ordinary differential equations.

The present book is designed to fill that need. Although the magnitude of the subject and the difficulties of its more advanced phases make impossible any attempt at completeness, it is believed that the treatment is sufficiently comprehensive to meet most requirements. Of necessity, major consideration has been given to the formal aspects of the subject, but its applications to geometry, physics, and engineering have also received their share of attention.

It will be found that the book is sufficiently adaptable to be used in several types of courses. Thus, methods of solving certain types of equations may be excluded without loss of continuity, and geometric and physical applications may be selected as desired.

Best results and maximum profit will be obtained if this book is made the basis of a course which follows a semester's course in ordinary differential equations. However, it is possible to use it following a calculus course which includes the material given, for example, in the final chapter of the author's "Calculus."

To augment previous studies of ordinary differential equations, especially for review and reference purposes, methods of solving such equations are treated in Chapter I. Since this material is intended only for the purposes stated, it is briefly presented, but each method is illustrated by means of an example, and 160 exercises for the student are included.

In addition, because two common difficulties in the study of partial differential equations are connected with the techniques of partial differentiation and solid analytic geometry, Chapter II has been inserted for further review and reference when needed. This chapter also contains numerous illustrative examples and 95 exercises.

Examination of the Table of Contents will quickly show the teacher what portions of Chapters I and II need be considered, by each class, before entering the study of partial differential equations.

Chapter III, the first concerned with the new subject itself, is designed to show the various ways in which partial differential equations come into being. Formal elimination processes, geometric problems, and diverse physical applications are there discussed. These considerations not only serve to induce the subsequent examination of methods of solution, but also propound live problems that the student will want to complete.

The remaining chapters, with the exception of Chapter VI, deal with methods of solving different classes of partial differential equations and with geometric and physical problems solvable by the processes explained. Chapter VI, on the formation and properties of Fourier series, contains the material essential to certain portions of the following chapter, in which these series play a part.

Throughout, particular care has been taken to motivate the work, and to present each argument logically and clearly. Some of the methods treated, notably in Chapter VII, are believed to be new, as are the presentations of various classical methods.

In addition to abundant illustrative examples, there are altogether 815 exercises for the student. Answers to all exercises are included.

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FREDERIC H. MILLER

CONTENTS

CHAPTER I

Ordinary Differential Equations

CHAPTER II

Partial Differentiation and Space Geometry

CHAPTER III

Origins of Partial Differential Equations

CHAPTER IV

Linear Equations of First Order

CHAPTER V

Non-Linear Equations of First Order

CHAPTER VI

Fourier Series

CHAPTER VII

Linear Equations of Second and Higher Orders

CHAPTER VIII

Non-Linear Equations of Second Order

PARTIAL DIFFERENTIAL EQUATIONS

CHAPTER I

ORDINARY DIFFERENTIAL EQUATIONS

This chapter is devoted to a summary of the methods of solving some frequently occurring types of ordinary differential equations. It is intended that this material shall serve merely for review and for reference in connection with the later work on partial differential equations. Accordingly, the underlying theory and motivations will be omitted, only the techniques and illustrative examples being given. For more detailed discussions, the student is referred to the various textbooks on ordinary differential equations or to those portions of numerous calculus textbooks which treat of differential equations.

1. Definitions. It will be recalled that a solution of an ordinary differential equation is any functional relation between the variables that identically satisfies the equation. When a solution contains, in addition to the variables, a number of essential arbitrary constants equal to the order of the differential equation, that solution is called the general solution, or complete primitive. Solutions obtained from the general solution by assigning specific values to one or more of the arbitrary constants are called particular solutions. Occasionally there exist so-called singular solutions, not obtainable from the general solution; these are briefly discussed in Arts. 7–8.

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

2. Variables separable. If, in the typical differential equation of first order,

$$\frac{dy}{dx} = f(x, y), \tag{1}$$

it is possible to separate the variables so that the equation assumes the form

$$M(x)\, dx + N(y)\, dy = 0, \tag{2}$$

integration at once yields the general solution,

$$\int M(x)\,dx + \int N(y)\,dy = c, \tag{3}$$

where c is the arbitrary constant of integration.

Example. Find the general solution of the equation

$$\frac{dy}{dx} = \frac{2xy + 3y}{x^2y - x^2}.$$

Solution. Here we have

$$x^2(y - 1)\,dy = y(2x + 3)\,dx,$$

$$\frac{y - 1}{y}\,dy = \frac{2x + 3}{x^2}\,dx,$$

$$y - \log y = 2\log x - \frac{3}{x} + c.$$

3. Integrable combinations. After a differential equation has been written in differential form, and the variables are not separable, it sometimes happens that combinations of terms, involving both variables and their differentials, are the exact differentials of certain expressions. If the entire equation is made up of such combinations and individual terms each of which involves only one variable and its differential, the equation is integrable by combinations.

Example 1. Find the general solution of the equation

$$\frac{dy}{dx} = \frac{y}{4y - x}. \tag{1}$$

Solution. If we write the given equation in the form

$$x\,dy + y\,dx - 4y\,dy = 0, \tag{2}$$

we see that the combination $x\,dy + y\,dx$ is the exact differential of the product xy. Since the remaining term, $-4y\,dy$, is also integrable, we get upon integration,

$$xy - 2y^2 = c. \tag{3}$$

An equation such as (2), in which the entire differential expression equated to zero is the exact differential of some function of x and y, is an exact differential equation. In some cases it is possible to multiply a given non-exact equation throughout by an integrating factor which renders the equation exact.

Example 2. Find the general solution of the equation

$$\frac{dy}{dx} = \frac{y}{4y + x}. \tag{4}$$

Solution. In differential form, this equation becomes

$$y\,dx - x\,dy - 4y\,dy = 0. \tag{5}$$

Here $1/y^2$ serves as an integrating factor, for

$$\frac{y\,dx - x\,dy}{y^2} - \frac{4\,dy}{y} = 0 \tag{6}$$

is exact. Hence we get

$$\frac{x}{y} - 4 \log y = c. \tag{7}$$

EXERCISES

Find the general solution of each of the following differential equations.

1. $\dfrac{dy}{dx} = \dfrac{2x}{y + 3}.$

2. $\dfrac{dy}{dx} = \dfrac{x - 3xy}{y}.$

3. $\dfrac{dy}{dx} = \dfrac{\cos y}{2 - x}.$

4. $\dfrac{dy}{dx} = \sqrt{\dfrac{1 - y^2}{1 - x^2}}.$

5. $\dfrac{dy}{dx} = xe^{x+y}.$

6. $\dfrac{dy}{dx} = \dfrac{y \log y}{3x}.$

7. $\dfrac{dy}{dx} = \sqrt{2 - x + 2y - xy}.$

8. $\dfrac{dy}{dx} = \cos(x + y) - \cos(x - y).$

9. $\dfrac{dy}{dx} = -\dfrac{3x + y}{x}.$

10. $\dfrac{dy}{dx} = \dfrac{y}{3y^2 - x}.$

11. $\dfrac{dy}{dx} = \dfrac{y - 6x^3}{x}.$

12. $\dfrac{dy}{dx} = \dfrac{x^2y^2 - y}{x}.$

13. $\dfrac{dy}{dx} = \dfrac{x^2 + y^2 + y}{x}.$

14. $\dfrac{dy}{dx} = \dfrac{x^2 - y^2 + y}{x}.$

15. $\dfrac{dy}{dx} = \dfrac{ye^x}{y - e^x}.$

16. $\dfrac{dy}{dx} = -\dfrac{2x + y \cos x}{\sin x}.$

17. $\dfrac{dy}{dx} = \dfrac{2x^3 + y^2}{2xy}.$

18. $\dfrac{dy}{dx} = \dfrac{\tan y - 3x^2}{x \sec^2 y}.$

19. $\dfrac{dy}{dx} = \dfrac{ye^x}{e^x - 4y^2}.$

20. $\dfrac{dy}{dx} = \dfrac{y}{x(\log x + y^2)}.$

4. Linear equations of first order. The type form of linear differential equation of first order is

$$\frac{dy}{dx} + Py = Q, \tag{1}$$

where P and Q are functions of x only. This equation has as an integrating factor the function

$$e^{\int P\,dx}, \tag{2}$$

and the solution of (1) is

$$y = e^{-\int P\,dx} \int Qe^{\int P\,dx}\,dx + ce^{-\int P\,dx}. \tag{3}$$

Example. Find the general solution of the equation

$$\frac{dy}{dx} = \frac{x^5 + 3y}{x}.$$

Solution. Writing the given equation in type form,

$$\frac{dy}{dx} - \frac{3}{x}y = x^4,$$

we see that $P = -3/x$, $Q = x^4$. Hence $\int P\,dx = -3 \log x$, and the integrating factor is

$$e^{\int P\,dx} = e^{-3\log x} = e^{\log x^{-3}} = x^{-3}.$$

Therefore

$$y = x^3 \int x^4 \cdot x^{-3}\,dx + cx^3 = \frac{x^5}{2} + cx^3.$$

In equation (1), which is linear in y and dy/dx, x and y are respectively considered as independent and dependent variables. When the roles played by x and y are reversed, the type form of first order linear equation is

$$\frac{dx}{dy} + Rx = S, \tag{4}$$

where now R and S are functions of y only.

An equation of the form

$$\frac{dy}{dx} + Py = Qy^n, \tag{5}$$

(Bernoulli's equation), where n is a constant other than zero or unity, and P and Q are functions of x only, may be transformed into a linear equation by means of the substitution $z = y^{1-n}$.

5. Homogeneous equations. If, in the equation

$$\frac{dy}{dx} = f(x, y), \tag{1}$$

the function $f(x, y)$ is such that $f(rx, ry) = f(x, y)$ identically for any quantity r, then $f(x, y)$ is said to be homogeneous of order zero and (1) is called a homogeneous differential equation.

The substitution

$$y = vx, \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}, \tag{2}$$

transforms a homogeneous equation into one in which the variables x and v are separable. Likewise, the substitution

$$x = vy, \qquad \frac{dx}{dy} = v + y\frac{dv}{dy}, \tag{3}$$

transforms a homogeneous equation into one in which the variables y and v are separable.

Example. Find the general solution of the equation

$$\frac{dy}{dx} = \frac{x^2 + y^2 + xy}{x^2}.$$

Solution. This is a homogeneous equation, so that we may make either substitution (2) or substitution (3). Choosing (2), we get

$$v + x\frac{dv}{dx} = 1 + v^2 + v,$$

$$\frac{dv}{1 + v^2} = \frac{dx}{x},$$

$$\arctan v = \log x + c,$$

$$v = \frac{y}{x} = \tan(\log x + c),$$

$$y = x \tan(\log x + c).$$

An equation of the form

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}, \tag{4}$$

where $a_1b_2 \neq a_2b_1$, may be transformed into a homogeneous differential equation

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y} \tag{5}$$

by means of the substitutions $x = X + h$, $y = Y + k$, in which (h, k) is the point of intersection of the lines

$$a_1x + b_1y + c_1 = 0, \qquad a_2x + b_2y + c_2 = 0. \tag{6}$$

An equation of the form (4), with $a_1b_2 = a_2b_1$, may be transformed into a differential equation in which the variables are separable by means of the substitution $z = a_1x + b_1y$.

EXERCISES

In Exercises 1–20, find the general solution of each differential equation:

1. $\dfrac{dy}{dx} = e^x + 2y.$

2. $\dfrac{dy}{dx} = 2x - y.$

3. $\dfrac{dy}{dx} = \sin 3x + 3y.$

4. $\dfrac{dy}{dx} = \dfrac{1 - 4xy}{x^2}.$

5. $\dfrac{dy}{dx} = \dfrac{\log x - 3x^3y}{x^4}.$

6. $\dfrac{dy}{dx} = \dfrac{x + y \log x}{x \log x}.$

7. $\dfrac{dy}{dx} = \dfrac{2}{y - x}.$

8. $\dfrac{dy}{dx} = \dfrac{1}{x \tan y + \sec y}.$

9. $\dfrac{dy}{dx} = \dfrac{y^2 - 2e^{-x}}{2y}.$

10. $\dfrac{dy}{dx} = 4y - xy^2.$

11. $\dfrac{dy}{dx} = \dfrac{xy + 4y^2}{x^2}.$

12. $\dfrac{dy}{dx} = \dfrac{y^3 - 3x^3}{xy^2}.$

13. $\dfrac{dy}{dx} = \dfrac{y}{x} + \sec \dfrac{y}{x}.$

14. $\dfrac{dy}{dx} = \dfrac{\sqrt{x^2 + y^2} + y}{x}.$

15. $\dfrac{dy}{dx} = \dfrac{1}{\log (y/x)} + \dfrac{y}{x}.$

16. $\dfrac{dy}{dx} = 1 + \cos \dfrac{y}{x} + \dfrac{y}{x}.$

17. $\dfrac{dy}{dx} = \dfrac{x^2 + xy + y^2}{xy}.$

18. $\dfrac{dy}{dx} = \dfrac{y}{\sqrt{y^2 - x^2} + x}.$

19. $\dfrac{dy}{dx} = \dfrac{x - y - 6}{x + y - 2}.$

20. $\dfrac{dy}{dx} = \dfrac{x - 2y + 3}{2x - 4y - 5}.$

For each of the differential equations of Exercises 21–26, find a particular solution satisfying the given condition.

21. $\dfrac{dy}{dx} = \dfrac{3x - y}{x}$; $y = 0$ for $x = 2$.

22. $\dfrac{dy}{dx} = \dfrac{2x^3 + y}{2x}$; $y = \dfrac{8}{5}$ for $x = 4$.

23. $\dfrac{dy}{dx} = 2\cos x + y \tan x$; $y = 1$ for $x = \pi$.

24. $\dfrac{dy}{dx} = 5y + e^{3x}$; $y = 6$ for $x = 0$.

25. $\dfrac{dy}{dx} = \dfrac{2x - y}{2y - x}$; $y = 0$ for $x = 1$.

26. $\dfrac{dy}{dx} = \dfrac{6y^3 - xy^2}{x^3}$; $y = 1$ for $x = 1$.

27. If $2\sqrt{x}(dy - dx) + y\,dx = 0$, and $y = 0$ for $x = 0$, find y for $x = \log^2 2$.

28. If $x^2\,dy + (xy - x^2 - y^2)\,dx = 0$, and $y = 2e$ for $x = e$, find y for $x = 1$.

29. If $dy = (e^{2x} + 3y)\,dx$, and $y = 5$ for $x = 0$, find the minimum value of y.

30. If $x\,dy + (\log x - 1 - y)\,dx = 0$, and $y = -3$ for $x = 1$, find the maximum value of y.

6. Equations of degree higher than the first. Among the equations of first order but of degree higher than unity in dy/dx, the following three types may be attacked by elementary methods: (*a*) equations solvable for dy/dx; (*b*) equations solvable for y; (*c*) equations solvable for x.

For convenience, denote dy/dx by p.

(*a*) When a differential equation of degree m $(m > 1)$ is solvable for p, such algebraic solution leads in general to m equations of the first degree, to which the methods of Arts. 2–5 may sometimes be applied. If $F_1(x, y, c_1) = 0$, $F_2(x, y, c_2) = 0$, $\cdots$, $F_m(x, y, c_m) = 0$ are the solutions of these m equations, the general solution of the original equation of degree m is expressed as

$$F_1(x, y, c)\cdot F_2(x, y, c) \cdots F_m(x, y, c) = 0. \tag{1}$$

Example 1. Find the general solution of the equation

$$2y\left(\frac{dy}{dx}\right)^2 + (6x^2y - 1)\frac{dy}{dx} - 3x^2 = 0. \tag{2}$$

Solution. The quadratic $2yp^2 + (6x^2y - 1)p - 3x^2 = 0$ may be solved for p to yield

$$p = \frac{dy}{dx} = \frac{1}{2y}, \qquad p = \frac{dy}{dx} = -3x^2. \tag{3}$$

These two equations have the respective solutions $y^2 - x + c_1 = 0$ and $y + x^3 + c_2 = 0$ (Art. 2). Hence the general solution of the given equation is $(y^2 - x + c)(y + x^3 + c) = 0$, or

$$x^3y^2 - x^4 + y^3 - xy + c(x^3 + y^2 - x + y) + c^2 = 0. \tag{4}$$

(*b*) When a differential equation is solvable for y, the solved form may be differentiated with respect to x to yield a new equation in the variables x and p; this will be a differential equation of the first order and first degree to which one of the methods of Arts. 2–5 may apply.

Example 2. Find the general solution of the equation

$$\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0. \tag{5}$$

Solution. Differentiating the equation

$$y = 2xp + p^2 \tag{6}$$

with respect to x, we get

$$p = 2x\frac{dp}{dx} + 2p + 2p\frac{dp}{dx},$$

or

$$2(x + p)\,dp + p\,dx = 0.$$

This equation may be solved as a homogeneous equation (Art. 5). However, we may solve it more easily by use of the integrating factor p. This gives us

$$2xp\,dp + p^2\,dx + 2p^2\,dp = 0,$$

$$xp^2 + \frac{2p^3}{3} = \frac{c}{3},$$

$$3xp^2 + 2p^3 = c,$$

$$x = \frac{c - 2p^3}{3p^2}. \tag{7}$$

Inserting this expression for x in equation (6), we find

$$y = \frac{2c - p^3}{3p}. \tag{8}$$

Equations (7) and (8) together serve as parametric equations of the family of integral curves, p playing the role of parameter. In this instance we may also eliminate p from (7) and (8) to obtain the rectangular equation

$$3x^2y^2 + 4y^3 - 4cx^3 - 6cxy - c^2 = 0.$$

(c) When a differential equation is solvable for x, the solved form may be differentiated with respect to y, setting $dx/dy = 1/p$. The resulting equation in y and p, of the first order and first degree, may again be attacked by one of the standard methods.

Example 3. Find the general solution of the equation of Example 2 by another method.

Solution. Equation (5) of Example 2 may be solved for x to yield

$$x = \frac{y - p^2}{2p}. \tag{9}$$

Differentiating with respect to y, we get

$$\frac{1}{p} = \frac{p\left(1 - 2p\dfrac{dp}{dy}\right) - (y - p^2)\dfrac{dp}{dy}}{2p^2},$$

whence there is found

$$(p^2 + y)\,dp + p\,dy = 0.$$

This is integrable by combinations; we find

$$\frac{p^3}{3} + yp = \frac{c}{3},$$

$$y = \frac{c - p^3}{3p}. \tag{10}$$

Equation (10) is equivalent to equation (8), Example 2. From (9) and (10) we also have

$$x = \frac{c - 4p^3}{6p^2}, \tag{11}$$

which is equivalent to (7).

7. Singular solutions. In all of the preceding discussion we have been concerned with the problem of finding the general solution of a given differential equation. As remarked in Art. 1, however, there sometimes exists a singular solution, which does not belong to the general solution.

A singular solution, when existent, may be regarded geometrically as the envelope of the family of curves constituting the general solution. Accordingly, if the general solution of a certain differential equation has been found to be

$$F(x, y, c) = 0, \tag{1}$$

a possible singular solution may be sought by the usual process for finding the envelope of the family of curves (1) (see Art. 25). This consists of differentiating (1) partially with respect to the parameter c, yielding the relation

$$\frac{\partial F(x, y, c)}{\partial c} = 0, \tag{2}$$

and then eliminating c between equations (1) and (2).

The result of this elimination, called the c-discriminant locus, may contain the envelope when there is one, but it may also lead to other loci, such as cusp- and node-loci. It is therefore necessary to determine by actual test whether the resultant equation is or is not a solution of the original differential equation.

Instead of seeking an envelope in the c-discriminant, it is sometimes more convenient to deal with the differential equation itself. If

$$G(x, y, p) = 0, \tag{3}$$

where $p = dy/dx$, is the differential equation, we may differentiate this partially with respect to p, getting an equation

$$\frac{\partial G(x, y, p)}{\partial p} = 0, \tag{4}$$

and then eliminate p from (3) and (4). If an envelope exists, it may be present in the resulting p-discriminant, but again various extraneous loci may appear. When a relation so found satisfies equation (3), it represents an envelope and singular solution, but otherwise it does not.

Example. Find the general solution of the equation

$$4(x - 2y - 1)\left(\frac{dy}{dx}\right)^2 + 4\frac{dy}{dx} - 1 = 0, \tag{5}$$

and show that it has a singular solution.

Solution. This differential equation may conveniently be solved by either of the methods of Art. 6 (*b*)–(*c*). Choosing the latter procedure, we have

$$x = 2y + 1 - \frac{1}{p} + \frac{1}{4p^2}, \tag{6}$$

$$\frac{1}{p} = 2 + \left(\frac{1}{p^2} - \frac{1}{2p^3}\right)\frac{dp}{dy},$$

$$2p^2(1 - 2p) = (2p - 1)\frac{dp}{dy}. \tag{7}$$

Evidently $p = \frac{1}{2}$ satisfies this equation. Inserting this value of p in (6), we get

$$x = 2y, \tag{8}$$

which is consistent with the relation $p = \frac{1}{2}$. Hence the straight line (8) is a solution of the given differential equation (5).

After extraction of the factor $1 - 2p$ from (7), we obtain

$$2p^2 = -\frac{dp}{dy},$$

$$dy = -\frac{dp}{2p^2},$$

$$y = \frac{1}{2p} + c. \tag{9}$$

Combining (9) with (6), there is found

$$x = 2c + 1 + (y - c)^2, \tag{10}$$

a family of parabolas.

Using either the c-discriminant obtained from (10), or the p-discriminant from (5), it is easily verified that the line (8) is the envelope of the parabolas (10). Thus (10) is the general solution and (8) is the singular solution of (5).

Neither of the differential equations of the examples of Art. 6 possesses a singular solution. In each case, the c- and p-discriminants yield loci which do not satisfy the corresponding differential equation. In Chapter II, Art. 25, there is given a sufficient condition for the existence of an envelope to a family of curves.

8. The Clairaut and d'Alembert equations. The Clairaut equation is a differential equation of the form

$$y = xp + \phi(p), \tag{1}$$

where $p = dy/dx$. Differentiating with respect to x, we get

$$p = x\frac{dp}{dx} + p + \phi'(p)\frac{dp}{dx},$$

$$[x + \phi'(p)]\frac{dp}{dx} = 0. \tag{2}$$

This equation is satisfied in two ways. If $dp/dx = 0$, then $p = c$, a constant, and (1) gives us

$$y = cx + \phi(c), \tag{3}$$

a family of straight lines. This is the general solution, obtainable immediately from (1) by replacing p by c.

On the other hand, (2) is satisfied if $x + \phi'(p) = 0$. Since this is the equation obtained by differentiating (1) partially with respect to p, elimination of p between (1) and that derived equation yields the p-discriminant locus.

Likewise, partial differentiation of (3) with respect to c, which gives us $x + \phi'(c) = 0$, and elimination of c between this equation and (3), yields the c-discriminant locus. Evidently the two discriminant loci are identical; both represent the envelope of the lines (3) and the singular solution of (1).

Example 1. Find the curve such that the algebraic sum of the intercepts of its tangent line on the coordinate axes is a constant a.

Solution. The equation of the tangent line at any point (x, y) of the curve is

$$Y - y = p(X - x),$$

where $p = dy/dx$ is the slope at that point. Then the x- and y-intercepts of the tangent line are respectively $x - y/p$ and $y - xp$, and we are to have

$$x - \frac{y}{p} + y - xp = a,$$

or

$$y = xp + \frac{ap}{p - 1}. \tag{4}$$

This is an equation of Clairaut's form. Hence the general solution is the family of lines

$$y = cx + \frac{ac}{c - 1}. \tag{5}$$

The envelope of these lines is found to be

$$x^2 - 2xy + y^2 - 2ax - 2ay + a^2 = 0, \tag{6}$$

a parabola tangent to the coordinate axes at $(a, 0)$ and $(0, a)$ and with its axis bisecting the first quadrant. This parabolic envelope (6) of the lines (5), which is the singular solution of the differential equation (4), is the desired result.

The d'Alembert equation is of the form

$$y = x\psi(p) + \phi(p); \tag{7}$$

it has as a special case the Clairaut equation, when $\psi(p) = p$. Differentiating (7) with respect to x, we get

$$p = x\psi'(p)\,\frac{dp}{dx} + \psi(p) + \phi'(p)\,\frac{dp}{dx},$$

or

$$p - \psi(p) = [x\psi'(p) + \phi'(p)]\,\frac{dp}{dx}. \tag{8}$$

If the equation

$$p - \psi(p) = 0 \tag{9}$$

has a real root r, so that $dp/dx = dr/dx = 0$, equation (8) will be satisfied. Consequently there may be one or more solutions of (7) of the form

$$y = \psi(r)x + \phi(r). \tag{10}$$

Such relations will in general be singular solutions; sometimes they will be particular solutions obtainable from the general solution.

To find the general solution of (7), we may write (8) in the form

$$\frac{dx}{dp} + \frac{\psi'(p)}{\psi(p) - p}\,x = \frac{\phi'(p)}{p - \psi(p)}.$$

This is linear in x and dx/dp. Solving by the method of Art. 4, and combining the result with (7), we find the general solution sought.

Equation (5) of Art. 7 serves as an example of a d'Alembert equation, in which $\psi(p) = \frac{1}{2}$.

EXERCISES

Find the general solution of each of the following differential equations, in which p denotes dy/dx. Also determine any singular solutions these equations may possess.

1. $p^2 - (2x + y)p + 2xy = 0$.
2. $x^2p^2 - 2xyp - x^2 + 2y^2 = 0$.
3. $x^2p^2 + 2xyp - x^3 + y^2 = 0$.
4. $x^2p^2 - 2xyp + y^2 - x = 0$.
5. $p^2 + x - y = 0$.
6. $yp^2 - 2xp + y = 0$.
7. $xp^2 - yp - y = 0$.
8. $p^2 + 2p + y - 2x + 1 = 0$.
9. $y^2p^2 + 2xyp - y^2 + 1 = 0$.
10. $xp^2 - yp + 4 = 0$.
11. $3p^2 - 2p + x = 0$.
12. $p^2 - yp - e^x = 0$.
13. $xp^2 - yp + x = 0$.
14. $p^2 + xp - y = 0$.
15. $x^2p^2 + 2(4 - xy)p + y^2 = 0$.
16. $p^3 - xp + y = 0$.
17. $xp^2 + (1 - y)p - 1 = 0$.
18. $2xp^2 - (3x + 2y)p + 3y + 1 = 0$.
19. $xp^2 - (x + y)p - y = 0$.
20. $p^2 - xp + y - x = 0$.

DIFFERENTIAL EQUATIONS OF ORDER HIGHER THAN THE FIRST

9. Equations with one variable absent. Apart from linear equations, two types of differential equations of common occurrence are those in which (*a*) the dependent variable is lacking; (*b*) the independent variable is missing.

(*a*) An *n*th order differential equation of the form

$$F\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}\right) = 0, \tag{1}$$

which does not contain the dependent variable y explicitly, may be transformed into an equation of order $n - 1$ by means of the substitution

$$p = \frac{dy}{dx} \tag{2}$$

together with the derived relations

$$\frac{dp}{dx} = \frac{d^2y}{dx^2}, \cdots, \frac{d^{n-1}p}{dx^{n-1}} = \frac{d^ny}{dx^n}. \tag{3}$$

When $n = 2$, the transformed equation of first order in dp/dx is often solvable by one of the methods previously discussed. If the differential equation is accompanied by given conditions, it is well to evaluate the constants of integration as they arise.

Example 1. Find the solution of the equation

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 8x^3,$$

which is such that $y = 5$ and $dy/dx = -2$ for $x = 1$.

Solution. Making the substitutions (2) and (3), we have the equation

$$x\frac{dp}{dx} - p = 8x^3.$$

This may be integrated either by combinations (Art. 3) or as a linear equation (Art. 4). By either method, we get

$$\frac{p}{x} = 4x^2 + c_1.$$

Using the fact that $p = dy/dx = -2$ for $x = 1$, we find $c_1 = -6$, whence

$$p = \frac{dy}{dx} = 4x^3 - 6x,$$

$$y = x^4 - 3x^2 + c_2.$$

Since $y = 5$ for $x = 1$, $c_2 = 7$, and the desired solution is

$$y = x^4 - 3x^2 + 7.$$

If, in an equation of type (1), not only y but its first and possibly other derivatives are lacking, and two or more derivatives of higher order appear, the procedure may be shortened by setting p equal to the derivative of lowest order present. If the equation contains only x and d^ny/dx^n, the equation may be solved by n successive integrations.

(*b*) An equation of the form

$$G\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}\right) = 0, \tag{4}$$

which does not contain the independent variable x explicitly, may be transformed into an equation of order $n - 1$ by means of substitution (2) together with the relations expressing d^2y/dx^2, d^3y/dx^3, $\cdots$ in terms of p and derivatives of p with respect to y; thus,

$$\frac{d^2y}{dx^2} = p\frac{dp}{dy}, \qquad \frac{d^3y}{dx^3} = p^2\frac{d^2p}{dy^2} + p\left(\frac{dp}{dy}\right)^2, \cdots. \tag{5}$$

Example 2. Find the solution of the equation

$$2y\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 - 3y\frac{dy}{dx} = 0,$$

which is such that $y = 2$ and $dy/dx = \frac{3}{2}$ for $x = 0$.

Solution. Substituting $p = dy/dx$, $p\,dp/dy = d^2y/dx^2$, we get

$$2yp\frac{dp}{dy} + 2p^2 - 3yp = 0.$$

By the conditions of the problem, p does not vanish identically, and therefore

$$2y\,dp + 2p\,dy - 3y\,dy = 0,$$

whence

$$2yp - \frac{3y^2}{2} = c_1.$$

Since $y = 2$ when $p = \frac{3}{2}$, $c_1 = 0$, and

$$p = \frac{dy}{dx} = \frac{3y}{4}.$$

Hence we get

$$\frac{dy}{y} = \frac{3}{4}\,dx,$$

$$\log y = \tfrac{3}{4}x + c_2.$$

Since $y = 2$ for $x = 0$, $c_2 = \log 2$, and

$$\log y - \log 2 = \tfrac{3}{4}x,$$

$$y = 2e^{3x/4}.$$

If equation (4) is lacking dy/dx, and possibly other derivatives, as well as x, but two or more derivatives of higher order appear, we may advantageously set p equal to the derivative of lowest order present.

EXERCISES

Find a particular solution of each of the differential equations in Exercises 1-14, subject to the given conditions.

1. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = e^x$; $y = 3$, $\dfrac{dy}{dx} = 2$, for $x = 0$.

2. $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 8x^3$; $y = \dfrac{5}{2}$, $\dfrac{dy}{dx} = 4$, for $x = 1$.

3. $x^3\dfrac{d^2y}{dx^2} - 3x^2\dfrac{dy}{dx} = 5$; $y = 7$, $\dfrac{dy}{dx} = -2$, for $x = -1$.

4. $x\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 1$; $y = -\pi$, $\dfrac{dy}{dx} = 0$, for $x = 2$.

5. $x^2\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 2x\dfrac{dy}{dx}$; $y = -8$, $\dfrac{dy}{dx} = 2$, for $x = 1$.

6. $x\dfrac{d^3y}{dx^3} = 4\dfrac{d^2y}{dx^2}$; $y = 2$, $\dfrac{dy}{dx} = -3$, $\dfrac{d^2y}{dx^2} = 10$, for $x = -1$.

7. $\dfrac{d^2y}{dx^2} = 4y$; $y = -2$, $\dfrac{dy}{dx} = 16$, for $x = 0$.

8. $\dfrac{d^2y}{dx^2} = -9y$; $y = 1$, $\dfrac{dy}{dx} = -6$, for $x = \pi$.

9. $\dfrac{d^2y}{dx^2} = 6y^2$; $y = 1$, $\dfrac{dy}{dx} = 2$, for $x = 0$.

10. $\dfrac{d^2y}{dx^2} = -\dfrac{4}{y^2}$; $y = 2$, $\dfrac{dy}{dx} = -2$, for $x = \dfrac{1}{3}$.

11. $\dfrac{d^2y}{dx^2} = y^2\dfrac{dy}{dx}$; $y = \sqrt{3}$, $\dfrac{dy}{dx} = \sqrt{3}$, for $x = -1$.

12. $2y\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$; $y = 2$, $\dfrac{dy}{dx} = -4$, for $x = -6$.

13. $\dfrac{dy}{dx}\dfrac{d^3y}{dx^3} = \left(\dfrac{d^2y}{dx^2}\right)^2$; $y = -3$, $\dfrac{dy}{dx} = 2$, $\dfrac{d^2y}{dx^2} = 4$, for $x = -1$.

14. $\dfrac{dy}{dx}\dfrac{d^3y}{dx^3} + 3y^2\left(\dfrac{dy}{dx}\right)^3 = \left(\dfrac{d^2y}{dx^2}\right)^2$; $y = 1$, $\dfrac{dy}{dx} = -\dfrac{1}{4}$, $\dfrac{d^2y}{dx^2} = \dfrac{1}{4}$, for $x = 4$.

Find the general solution of each of the differential equations in Exercises 15–20.

15. $\dfrac{d^2y}{dx^2} = k^2y.$

16. $\dfrac{d^2y}{dx^2} = -k^2y.$

17. $\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^3 = 4\left(\dfrac{d^2y}{dx^2}\right)^2.$

18. $y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 + 1 = 0.$

19. $x\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^3 = \dfrac{dy}{dx}.$

20. $\left[y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2\right]^2 = 12y\dfrac{dy}{dx}.$

10. Linear equations of higher order. A linear differential equation of order n is an equation of the form

$$P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_{n-1}\frac{dy}{dx} + P_ny = Q, \tag{1}$$

where P_0, P_1, $\cdots$, P_n, and Q are functions of the independent variable only, $P_0 \neq 0$. If $Q = 0$, equation (1) is called a homogeneous linear equation; if $Q \neq 0$, (1) is non-homogeneous. When $Q \neq 0$, the equation

$$P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_{n-1}\frac{dy}{dx} + P_ny = 0, \tag{2}$$

obtained from (1) by replacing Q by zero, is said to be the reduced equation corresponding to (1).

If $y = Y_c(x)$ is the general solution of equation (2), containing n arbitrary constants, then Y_c is called the complementary function of (1). If $y = Y_p(x)$ is any solution of the non-homogeneous equation (1), then Y_p is called a particular integral of (1). The general solution of the non-homogeneous equation (1) is made up of the sum of the complementary function and a particular integral.

If $y = u_1(x)$ is any solution of the homogeneous equation (2), then $y = cu_1$, where c is an arbitrary constant, is also a solution. If $y = u_1(x)$ and $y = u_2(x)$ are two solutions of (2), then $y = c_1u_1 + c_2u_2$, where c_1 and c_2 are arbitrary constants, is also a solution.

It is customary to use the symbol D to denote the operation of differentiation with respect to the independent variable x, and, more generally, to denote by D^r the operation of finding the rth derivative:

$$\mathrm{D}^r \equiv \frac{d^r}{dx^r} \quad (r = 1, 2, \cdots, n). \tag{3}$$

With this notation, equation (1) may be written briefly as

$$\phi(\mathrm{D})y = Q, \tag{4}$$

where $\phi(\mathrm{D})$ is the operator

$$\phi(\mathrm{D}) \equiv P_0\mathrm{D}^n + P_1\mathrm{D}^{n-1} + \cdots + P_{n-1}\mathrm{D} + P_n. \tag{5}$$

If $\phi(\mathrm{D})$ is any linear differential operator of the form (5), the following relations hold:

$$\phi(\mathrm{D})e^{cx} = \phi(c)e^{cx}, \tag{6}$$

where c is any constant;

$$\phi(\mathrm{D})(e^{cx}u) = e^{cx}\phi(\mathrm{D} + c)u, \tag{7}$$

where u is any function of x and c is a constant. Equation (7), which is sometimes called the shifting formula, has numerous applications in the theory and in the processes of solving linear equations.

Examples. From relations (6) and (7) we have

$$(\mathrm{D}^2 - 3\mathrm{D} + 5)\, e^{2x} = (2^2 - 3\cdot 2 + 5)\, e^{2x} = 3e^{2x},$$
$$(x^2\mathrm{D}^2 - 2x\mathrm{D})\, x^3 = x^2\cdot 6x - 2x\cdot 3x^2 = 0,$$
$$(2x\mathrm{D}^2 - 3)(e^{-x}\sin x) = e^{-x}\,[2x(\mathrm{D} - 1)^2 - 3]\sin x$$
$$= e^{-x}\,(2x\mathrm{D}^2 - 4x\mathrm{D} + 2x - 3)\sin x,$$
$$(\mathrm{D} - c)^r(e^{cx}u) = e^{cx}\mathrm{D}^r u.$$

11. The complementary function. In this article, and in the following one concerned with particular integrals, we shall consider only those linear equations in which the coefficients of the operator $\phi(\mathrm{D})$ are real constants.

The process of finding the complementary function of a linear equation with real constant coefficients may be summed up in the following statement.

Let $r_1, r_2, \cdots, r_n$ be the roots of the algebraic equation

$$a_0m^r + a_1m^{r-1} + \cdots + a_{n-1}m + a_n = 0, \tag{1}$$

auxiliary to the linear homogeneous equation of nth order,

$$\phi(\mathrm{D})y \equiv (a_0\mathrm{D}^n + a_1\mathrm{D}^{n-1} + \cdots + a_{n-1}\mathrm{D} + a_n)y = 0, \tag{2}$$

where $a_0, a_1, \cdots, a_n$ are any real constants. (*a*) If $r_1, r_2, \cdots, r_n$ are real and all different, the general solution of (2) is

$$y = c_1e^{r_1x} + c_2e^{r_2x} + \cdots + c_ne^{r_nx},$$

where $c_1, c_2, \cdots, c_n$ are arbitrary constants. (*b*) If r is a p-fold real root of (1), the portion of the general solution of (2), corresponding to r, is

$$(c_1 + c_2x + \cdots + c_px^{p-1})e^{rx}.$$

(c) If conjugate complex roots $\alpha \pm i\beta$, where $i = \sqrt{-1}$, occur p times, the corresponding part of the general solution is

$$e^{\alpha x}[(c_1 + c_2x + \cdots + c_px^{p-1}) \sin \beta x + (c_1' + c_2'x + \cdots + c_p'x^{p-1}) \cos \beta x].$$

Example 1. Solve the equation

$$(D^3 - D^2 - 6D)y = 0.$$

Solution. The roots of the auxiliary equation $m^3 - m^2 - 6m = 0$ are $m = 0, 3, -2$. Hence the general solution of the given equation is

$$y = c_1 + c_2e^{3x} + c_3e^{-2x}.$$

Example 2. Solve the equation

$$(D^3 - 10D^2 + 25D)y = 0.$$

Solution. Since $m^3 - 10m^2 + 25m = m(m - 5)^2$, the roots of the auxiliary equation are $m = 0, 5, 5$. Therefore

$$y = c_1 + (c_2 + c_3x)e^{5x}$$

is the general solution sought.

Example 3. Solve the equation

$$(D^3 + D^2 + 3D - 5)y = 0.$$

Solution. Here the roots of the equation $m^3 + m^2 + 3m - 5 = 0$ are found to be $m = 1, -1 \pm 2i$. Consequently we have

$$y = c_1e^x + e^{-x}(c_2 \sin 2x + c_3 \cos 2x).$$

EXERCISES

Find the general solution of each of the differential equations in Exercises 1–14, in which D denotes d/dx.

1. $(D^2 - 9)y = 0$.
2. $(D^2 + 9)y = 0$.
3. $(D^2 + 5D)y = 0$.
4. $(D^2 - 3D - 28)y = 0$.
5. $(D^2 + 6D + 9)y = 0$.
6. $(D^3 + 6D^2)y = 0$.
7. $(D^3 - 6D^2 + 12D - 8)y = 0$.
8. $(D^3 - 8)y = 0$.
9. $(D^4 - 16)y = 0$.
10. $(D^4 - 8D^2 + 16)y = 0$.
11. $(D^4 - 3D^3 + 3D^2 - D)y = 0$.
12. $(16D^4 - 32D^3 + 24D^2 - 8D + 1)y = 0$.
13. $(D^4 + 4)y = 0$.
14. $(D^5 - 5D^3 - 36D)y = 0$.

Find a particular solution of each of the differential equations in Exercises 15–20 subject to the given conditions.

15. $(D^2 - 4)y = 0$; $y = -1$, $Dy = 10$, for $x = 0$.
16. $(D^2 + 16)y = 0$; $y = -3$, $Dy = 0$, for $x = \pi$.
17. $(3D^2 - 4D)y = 0$; $y = 5$, $Dy = 0$, for $x = 0$.
18. $(D^2 + 2D + 1)y = 0$; $y = 0$, $Dy = 2$, for $x = 0$.
19. $(D^3 + 3D^2 - 4D)y = 0$; $y = -1$, $Dy = -7$, $D^2y = 13$, for $x = 0$.
20. $(D^3 - D^2 + D - 1)y = 0$; $y = 1$, $Dy = -2$, $D^2y = 1$, for $x = 0$.

12. Particular integrals. Among the various methods of finding a particular integral of a linear non-homogeneous equation with constant coefficients, only two, of considerable power and generality, will be discussed here.

Let the differential equation under discussion be denoted, as before, by

$$\phi(D)y \equiv (a_0D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = Q, \qquad (1)$$

where $a_0, a_1, \cdots, a_n$ are constants and Q is a non-zero function of x. If Q consists of the sum of two or more terms, $Q = Q_1 + Q_2 + \cdots + Q_s$, the particular integral Y_p may be found as the sum of s terms, $Y_{p1} + Y_{p2} + \cdots + Y_{ps}$, where Y_{pj} is the portion corresponding to Q_j, that is, $\phi(D)\,Y_{pj} = Q_j$ identically $(j = 1, 2, \cdots, s)$. Thus a part of the particular integral, corresponding to a part of Q, may be determined by one method, and the rest of Y_p may be found by another method.

(*a*) *Method of undetermined coefficients.* If the right member Q of (1) consists entirely or in part of terms of the forms

$$Cx^qe^{cx}, \qquad Cx^qe^{\alpha x}\sin \beta x, \qquad Cx^qe^{\alpha x}\cos \beta x, \qquad (2)$$

where q is a positive integer or zero, c, α, and β are any real constants, and C is any constant, the corresponding part of the particular integral may be found by applying the following rule. For definiteness, we suppose Q to consist entirely of terms of the forms (2).

Write the variable parts of the terms of the right member Q and of all other terms obtainable by repeated differentiation of Q. When a term is a constant, write 1. Arrange all of these terms into groups such that all terms obtainable from a single term of Q appear in only one group. If no member of a specific group appears as a term in the complementary function Y_c, all members of that group are left intact; if any member of a group is a term of Y_c, all members of that group are multiplied by the lowest positive integral power of x that makes them all different from any term of Y_c. Then give each resulting member in all groups a literal constant coefficient and take the sum of the expressions thus obtained as the particular integral Y_p. Substitute this sum in the differential equation (1), match corresponding terms, and produce an identity by determining proper values of the literal coefficients.

Example 1. Solve the equation

$$(D^2 - 4D + 3)y = 6x - 11 + 8e^{-x}.$$

Solution. The complementary function is $Y_c = c_1e^x + c_2e^{3x}$. From the right member we get two groups,

$$x, 1 \qquad \text{and} \qquad e^{-x}.$$

Since no member of either group appears in Y_c, we take

$$Y_p = Ax + B + Ce^{-x}.$$

Substituting in the differential equation we find

$$Ce^{-x} - 4A + 4Ce^{-x} + 3Ax + 3B + 3Ce^{-x} = 6x - 11 + 8e^{-x},$$

whence we must have $A = 2$, $B = -1$, $C = 1$. Therefore $Y_p = 2x - 1 + e^{-x}$; and the general solution is

$$y = c_1e^x + c_2e^{3x} + 2x - 1 + e^{-x}.$$

Example 2. Solve the equation

$$(\mathrm{D}^3 + \mathrm{D}^2)y = 4 - 12e^{2x}.$$

Solution. Here $Y_c = c_1 + c_2x + c_3e^{-x}$. The right member leads to the two groups

$$1 \qquad \text{and} \qquad e^{2x}.$$

Now 1 is a term of the complementary function, and must be multiplied by x^2 to make it different from any term of Y_c. Since e^{2x} is not part of Y_c, this group is left intact. Consequently we take

$$Y_p = Ax^2 + Be^{2x},$$

and substitution gives us

$$8Be^{2x} + 2A + 4Be^{2x} = 4 - 12e^{2x},$$

so that $A = 2$ and $B = -1$. Hence $Y_p = 2x^2 - e^{2x}$, and the general solution is

$$y = c_1 + c_2x + c_3e^{-x} + 2x^2 - e^{2x}.$$

The shifting formula (Art. 10) can often be used to advantage in the determination of particular integrals, as illustrated in the following example.

Example 3. Solve the equation

$$(\mathrm{D}^2 - 4\mathrm{D} + 5)y = 2e^{2x}\sin x.$$

Solution. In this case we have $Y_c = e^{2x}(c_1\sin x + c_2\cos x)$. Since the right member yields the group

$$e^{2x}\sin x, \qquad e^{2x}\cos x,$$

both members of which appear in Y_c, we should have to take $Y_p = Axe^{2x}\sin x + Bxe^{2x}\cos x$. We may, however, avoid this rather cumbersome form by setting

$$y = ue^{2x}.$$

Then, by formula (7) of Art. 10,

$$(\mathrm{D}^2 - 4\mathrm{D} + 5)(ue^{2x}) = e^{2x}[(\mathrm{D} + 2)^2 - 4(\mathrm{D} + 2) + 5]u = e^{2x}(\mathrm{D}^2 + 1)u,$$

and the transformed equation is, after cancellation of the factor e^{2x},

$$(D^2 + 1)u = 2 \sin x.$$

For this, $U_c = c_1 \sin x + c_2 \cos x$, and we set

$$U_p = Ax \sin x + Bx \cos x.$$

Upon substitution, this gives us

$$-Ax \sin x + 2A \cos x - Bx \cos x - 2B \sin x + Ax \sin x + Bx \cos x = 2 \sin x,$$

$$A = 0, \qquad B = -1,$$

and

$$U_p = -x \cos x.$$

Consequently $u = c_1 \sin x + c_2 \cos x - x \cos x$, and since $y = ue^{2x}$, we get the general solution of the original equation,

$$y = e^{2x}(c_1 \sin x + c_2 \cos x - x \cos x).$$

(*b*) *Method of variation of parameters.* It was stated above that the method of undetermined coefficients applies only to those parts of the right member Q that are of certain forms. The method of variation of parameters, due to Lagrange, applies to all cases, even when the coefficients of the operator depend on x, provided only that the complementary function is known. However, Lagrange's method usually entails longer and more difficult computations and integrations, and is therefore best reserved for use in those problems to which the first method is not applicable.

Let the complementary function of the equation

$$\phi(D)y \equiv (P_0 D^n + P_1 D^{n-1} + \cdots + P_{n-1} D + P_n)y = Q \tag{3}$$

be

$$Y_c = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n. \tag{4}$$

Take as the general solution the expression obtained from Y_c by replacing the arbitrary constants by variable parameters:

$$y = V_1 u_1 + V_2 u_2 + \cdots + V_n u_n. \tag{5}$$

Differentiating (5), we get

$$y' = V_1 u_1' + V_2 u_2' + \cdots + V_n u_n' + V_1' u_1 + V_2' u_2 + \cdots + V_n' u_n.$$

Now set that portion of y' containing the first derivatives of the parameters equal to zero,

$$V_1' u_1 + V_2' u_2 + \cdots + V_n' u_n = 0, \tag{6$_1$}$$

so that y' reduces to

$$y' = V_1u_1' + V_2u_2' + \cdots + V_nu_n'. \tag{7_1}$$

Differentiate this, getting

$$y'' = V_1u_1'' + V_2u_2'' + \cdots + V_nu_n'' + V_1'u_1' + V_2'u_2' + \cdots + V_n'u_n',$$

and again set the portion containing the first derivatives of the parameters equal to zero,

$$V_1'u_1' + V_2'u_2' + \cdots + V_n'u_n' = 0, \tag{6_2}$$

so that

$$y'' = V_1u_1'' + V_2u_2'' + \cdots + V_nu_n''. \tag{7_2}$$

Treat similarly $y''', \cdots, y^{(n-1)}$, thereby getting, altogether, $n - 1$ linear algebraic equations in $V_1', V_2', \cdots, V_n'$, the last of which is

$$V_1'u_1^{(n-2)} + V_2'u_2^{(n-2)} + \cdots + V_n'u_n^{(n-2)} = 0. \tag{6_{n-1}}$$

This leaves

$$y^{(n-1)} = V_1u_1^{(n-1)} + V_2u_2^{(n-1)} + \cdots + V_nu_n^{(n-1)}, \tag{7_{n-1}}$$

whence

$$y^{(n)} = V_1u_1^{(n)} + \cdots + V_nu_n^{(n)} + V_1'u_1^{(n-1)} + \cdots + V_n'u_n^{(n-1)}. \tag{7_n}$$

Substitute from (5), $(7_1), \cdots, (7_n)$ in (3); since $\phi(\mathrm{D})u_j = 0$ ($j = 1, 2, \cdots, n$), we then get

$$V_1'u_1^{(n-1)} + V_2'u_2^{(n-1)} + \cdots + V_n'u_n^{(n-1)} = Q/P_0. \tag{6_n}$$

The n linear algebraic equations $(6_1), \cdots, (6_n)$ may then be solved for the derivatives $V_1', \cdots, V_n'$, the resulting functions integrated to get the parameters $V_1, \cdots, V_n$ themselves, and these values substituted in (5) to get the general solution of (3).

Example 4. Solve the equation

$$(\mathrm{D}^3 + \mathrm{D})y = \sec x.$$

Solution. The complementary function is $Y_c = c_1 + c_2 \sin x + c_3 \cos x$. Hence we set

$$y = V_1 + V_2 \sin x + V_3 \cos x.$$

Then

$$y' = V_2 \cos x - V_3 \sin x$$

provided that

$$V_1' + V_2' \sin x + V_3' \cos x = 0.$$

Consequently

$$y'' = -V_2 \sin x - V_3 \cos x$$

provided that

$$V_2' \cos x - V_3' \sin x = 0,$$

and

$$y''' = -V_2 \cos x + V_3 \sin x - V_2' \sin x - V_3' \cos x.$$

Substituting in the given differential equation, we get

$$-V_2' \sin x - V_3' \cos x = \sec x.$$

Solving for V_1', V_2', V_3', we find

$$V_1' = \sec x, \qquad V_2' = -\tan x, \qquad V_3' = -1,$$

whence

$$V_1 = \log (\sec x + \tan x) + c_1, \qquad V_2 = \log \cos x + c_2, \qquad V_3 = -x + c_3.$$

Therefore the general solution sought is

$$y = c_1 + c_2 \sin x + c_3 \cos x + \log (\sec x + \tan x) + \sin x \log \cos x - x \cos x.$$

EXERCISES

Find the general solution of each of the following differential equations, in which D denotes d/dx.

1. $(D^2 - 4)y = 12x^2 - 16x - 6.$
2. $(D^2 + D - 2)y = 2e^{-x} - 12.$
3. $(D^2 + D)y = (2x + 4)e^x.$
4. $(D^2 + 9)y = 5 \sin 2x.$
5. $(D^2 - 3D)y = 14e^{2x} - 9.$
6. $(D^2 + 2D - 8)y = 12xe^{2x}.$
7. $(D^2 + 1)y = 6 \sin x.$
8. $(D^3 - 9D)y = 8e^x - 90x.$
9. $(D^3 + D^2)y = 16 - e^{-x}.$
10. $(D^3 + 4D)y = -20 - 16 \sin 2x.$
11. $(D^4 - 1)y = 4 \sin x - 12e^x.$
12. $(D^2 + 1)y = 2 \tan x.$
13. $(D^2 + 4)y = 8 \cot 2x.$
14. $(D^2 - 2D + 1)y = 3e^x/x^2.$
15. $(D^2 + 4D + 5)y = 3e^{-2x} \cos 2x.$
16. $(D^2 + 4)y = 8 \sec 2x.$
17. $(D^2 + 2D + 10)y = 9e^{-x} \tan 3x.$
18. $(D^2 - 6D + 10)y = 5e^{3x} \csc x.$
19. $(D^2 + 1)y = 4 \sec 2x.$
20. $(D^3 - 4D)y = 8/(e^x + 1).$

13. The Euler linear equation. We now consider linear differential equations of the type

$$(b_0x^nD^n + b_1x^{n-1}D^{n-1} + \cdots + b_{n-1}xD + b_n)y = Q, \tag{1}$$

where the b's are constants, $b_0 \neq 0$, Q is any function of x, and $D \equiv d/dx$. Equations of type (1), variously called Euler, Cauchy, or homogeneous linear equations, may always be transformed into linear equations with constant coefficients by means of the substitution

$$x = e^z. \tag{2}$$

For, if we use D_z to denote differentiation with respect to z, we have

$$x^rD^ry = D_z(D_z - 1)(D_z - 2) \cdots (D_z - r + 1)y \tag{3}$$

$(r = 1, 2, \cdots, n)$. Substituting these expressions in (1), and at the same time replacing x by e^z in the right member Q, we get a linear equation with constant coefficients in the variables z and y. If we solve this equation by the methods of Arts. 11–12, and replace z by its equivalent, $\log x$, in the result, we have the solution of the given equation (1).

Example. Solve the equation

$$(x^3D^3 + 2x^2D^2 - 2xD)y = 2x - 4.$$

Solution. From relations (2) and (3) we get

$$[D_z(D_z - 1)(D_z - 2) + 2D_z(D_z - 1) - 2D_z]y = 2e^z - 4,$$

$$(D_z^3 - D_z^2 - 2D_z)y = 2e^z - 4.$$

The complementary function is readily found to be

$$Y_c = c_1 + c_2e^{-z} + c_3e^{2z},$$

and the particular integral is of the form

$$Y_p = Ae^z + Bz.$$

Substitution gives us $A = -1$, $B = 2$, and consequently

$$y = c_1 + c_2e^{-z} + c_3e^{2z} - e^z + 2z.$$

Replacing z by $\log x$, we get as the general solution of the given equation,

$$y = c_1 + \frac{c_2}{x} + c_3x^2 - x + 2\log x.$$

A more general form of linear equation, including Euler's equation as a special case, is the Lagrange form

$$[b_0(\alpha x+\beta)^nD^n+b_1(\alpha x+\beta)^{n-1}D^{n-1}+\cdots+b_{n-1}(\alpha x+\beta)D+b_n]y=Q, \quad (4)$$

in which the linear expression $\alpha x + \beta$, α and β constants, replaces the x in the operator of equation (1). The substitution

$$\alpha x + \beta = e^z \quad (5)$$

leading to the relations

$$(\alpha x + \beta)^rD^ry = \alpha^rD_z(D_z - 1)\cdots(D_z - r + 1)y \quad (6)$$

$(r = 1, 2, \cdots, n)$, serves to transform (4) into an equation with constant coefficients.

14. Simultaneous linear equations. Let there be given a system of m linear differential equations with constant coefficients, involving one independent variable x and m dependent variables $y_1, y_2, \cdots, y_m$. Using D to denote d/dx as before, write the system as

$$
\begin{aligned}
\phi_{11}(D)y_1 + \phi_{12}(D)y_2 + \cdots + \phi_{1m}(D)y_m &= Q_1, \\
\phi_{21}(D)y_1 + \phi_{22}(D)y_2 + \cdots + \phi_{2m}(D)y_m &= Q_2, \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\quad \cdot, \\
\phi_{m1}(D)y_1 + \phi_{m2}(D)y_2 + \cdots + \phi_{mm}(D)y_m &= Q_m,
\end{aligned} \tag{1}
$$

where the operators $\phi_{jk}(D)$ $(j, k = 1, 2, \cdots, m)$, of any orders, have constant coefficients and $Q_1, Q_2, \cdots, Q_m$ are functions of x.

To solve a system (1) for $y_1, y_2, \cdots, y_m$, the procedure is analogous to that used to solve a system of linear algebraic equations. By suitable operations upon the equations (1), eliminate all but one of the dependent variables and its derivatives. Then solve for this dependent variable by the methods of Arts. 11–12. The remaining y's may be obtained either independently by similar elimination processes or by using those already found. In either case, care must be taken that the solutions found for all m y's satisfy the original system (1).

Example. Solve the system of equations

$$(D + 2)y - (3D + 1)z = 8e^{2x} + 7 - x, \tag{2}$$

$$Dy - (2D + 4)z = 4e^{2x} - 2 - 4x. \tag{3}$$

Solution. In this problem, we note that subtraction of (3) from (2) yields a relation containing no derivatives of y, namely,

$$2y - (D - 3)z = 4e^{2x} + 9 + 3x. \tag{4}$$

Hence, if we first find z, y may be obtained from (4) without further integration.

Operating on (2) with D and on (3) with D + 2, in order to make the coefficients of y alike, we get

$$D(D + 2)y - (3D^2 + D)z = 16e^{2x} - 1,$$

$$(D + 2)Dy - (2D^2 + 8D + 8)z = 16e^{2x} - 8 - 8x.$$

Subtracting the upper equation from the lower one, we have

$$(D^2 - 7D - 8)z = -7 - 8x.$$

From this we readily get

$$z = c_1e^{-x} + c_2e^{8x} + x. \tag{5}$$

Substituting in (4), there is obtained

$$2y + c_1e^{-x} - 8c_2e^{8x} - 1 + 3c_1e^{-x} + 3c_2e^{8x} + 3x = 4e^{2x} + 9 + 3x,$$

whence

$$y = -2c_1e^{-x} + \tfrac{5}{2}c_2e^{3x} + 2e^{2x} + 5. \tag{6}$$

The general solution of the system (2)–(3) is thus given by the two relations (5) and (6).

A convenient test for the determination of the total number of arbitrary constants that should be present in the general solution of a system of equations is obtainable from the operational determinant of the system. If the degree in D of the determinant $|\phi_{jk}(D)|$ is s, there should be just s arbitrary constants in the general solution expressions for $y_1, y_2, \cdots, y_m$ taken together.

EXERCISES

Find the general solution of each of the differential equations in Exercises 1–10, in which $D \equiv d/dx$.

1. $(x^2D^2 + xD)y = 6 \log x - 1/x$.
2. $(xD^2 - 3D)y = 4x - 6$.
3. $(x^2D^2 + xD + 1)y = 3 \sin (2 \log x)$.
4. $(x^2D^2 - 2xD + 2)y = 2x - 8$.
5. $(x^2D^2 + xD + 4)y = 4 \cos (2 \log x)$.
6. $(x^2D^2 + 6xD + 6)y = 8(\log x + 1)/x^2$.
7. $(x^3D^3 - x^2D^2 - 3xD)y = 18 - 72 \log x$.
8. $(x^3D^3 + 3x^2D^2 + xD - 8)y = 60x^2 + 24$.
9. $[(2x + 1)^2D^2 - 2(2x + 1)D + 4]y = 48 \log (2x + 1) - 12 \log^2 (2x + 1)$.
10. $[(3x - 2)^2D^2 + 3(3x - 2)D + 1]y = 60x - 45$.

In Exercises 11–20, find the general solution of each system of differential equations, in which $D \equiv d/dx$.

11. $(D - 3)y - (2D + 1)z = 8e^{-x} - 5$, $2y + (D + 2)z = 10 - 4e^{-x}$.
12. $(3D - 1)y + (D + 4)z = 20x + 24$, $(2D + 11)y + Dz = 44x - 8$.
13. $(D^2 + 1)y + D^2z = 6e^{2x}$, $(D - 3)y + (D - 1)z = 6e^{2x}$.
14. $(2D + 5)y + (D - 2)z = 16 \sin x + 8 \cos x$, $(2D + 7)y + (D - 1)z = 22 \sin x + 7 \cos x$.
15. $(3D - 1)y + Dz = 4 + 18x - 3x^2$, $(2D^2 + 2)y + (D^2 + 1)z = 11 + 4x + 6x^2$.
16. $3Dy + 2z = 0$, $(D + 1)z + 3w = 15 - 3x$, $D^2y - 2w = -10$.
17. $Dy - 2z = 2$, $Dz - 2w = 2$, $Dw - 2y = 2$.
18. $D^2y - 2z + Dw = -6$, $y + (D - 1)z - Dw = 2e^x - 3$, $(D + 1)y + w = e^x$.
19. $xDy - (3xD - 5)z = 15 - 12x^2$, $y + (xD + 1)z = 10 + 22x^2$.
20. $(x^2D^2 + 1)y - 3xDz = (12 + 15 \log x)x^2$, $(xD - 4)y + 4z = (3 - 10 \log x)x^5$.

15. Series solutions. Up to this point, all of the differential equations considered have been of special types, solvable in finite form. When a differential equation is not, and cannot readily be transformed into, one of these particular types, it is sometimes desirable to find a solution in the form of an infinite series.

Maclaurin's series,

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_rx^r + \cdots, \tag{1}$$

is frequently assumed as a solution of the given equation, the numbers $a_0, a_1, a_2, \cdots$ being determined by substituting (1) in the equation and setting the complete coefficient of each power of x equal to zero.

A generalization of this method, embodying a series of the form

$$y = x^c(a_0 + a_1x + a_2x^2 + \cdots + a_rx^r + \cdots), \tag{2}$$

where the index c and the a's are to be determined, may be used when a simple power series (1) does not suffice.

This procedure is often used in connection with homogeneous linear differential equations, when it is known as the method of Frobenius. The equation obtained by setting equal to zero the total coefficient of the lowest power of x in the series found by substitution in the linear equation, is called the indicial equation, for it serves to determine permissible values of the index c.

Example. Apply the method of Frobenius to the equation

$$2x\frac{d^2y}{dx^2} + \frac{dy}{dx} + 2y = 0.$$

Solution. After substituting series (2), together with its first two derivatives, in the given equation, we get

$$\begin{aligned}
&2a_0c(c-1)x^{c-1} + 2a_1(c+1)cx^c + \cdots + 2a_r(c+r)(c+r-1)x^{c+r-1} + \cdots \\
&+a_0cx^{c-1} \quad + a_1(c+1)x^c \quad + \cdots + a_r(c+r)x^{c+r-1} \quad + \cdots \\
&\qquad\qquad + 2a_0x^c \qquad + \cdots + 2a_{r-1}x^{c+r-1} \qquad + \cdots = 0.
\end{aligned}$$

Setting the total coefficients of $x^{c-1}, x^c, \cdots, x^{c+r-1}, \cdots$ equal to zero, we have the indicial equation

$$a_0c(2c-1) = 0,$$

and the additional relations

$$a_1(c+1)(2c+1) + 2a_0 = 0, \cdots, \qquad a_r(c+r)(2c+2r-1) + 2a_{r-1} = 0, \cdots.$$

From the indicial equation there is found (assuming $a_0 \neq 0$, as we may),

$$c = 0 \quad \text{or} \quad c = \tfrac{1}{2}.$$

When $c = 0$,

$$a_1 = -2a_0, \qquad a_2 = -\tfrac{1}{3}a_1 = \tfrac{2}{3}a_0, \qquad a_3 = -\tfrac{2}{15}a_2 = -\tfrac{4}{45}a_0, \cdots,$$

and, in general,

$$a_r = -\frac{2a_{r-1}}{r(2r-1)} = \frac{2^2a_{r-2}}{r(2r-1)(r-1)(2r-3)} = \cdots = \frac{(-1)^r2^{2r}a_0}{(2r)!};$$

when $c = \frac{1}{2}$,

$$a_1 = -\frac{2}{3}a_0, \qquad a_2 = -\frac{a_1}{5} = \frac{2}{15}a_0, \qquad a_3 = -\frac{2}{21}a_2 = -\frac{4}{315}a_0, \cdots,$$

and, in general,

$$a_r = -\frac{2a_{r-1}}{r(2r+1)} = \frac{2^2 a_{r-2}}{r(2r+1)(r-1)(2r-1)} = \cdots = \frac{(-1)^r 2^{2r} a_0}{(2r+1)!}.$$

Corresponding to $c = 0$, we therefore have as one solution,

$$y = a_0\left[1 - 2x + \frac{2}{3}x^2 - \frac{4}{45}x^3 + \cdots + (-1)^r\frac{2^{2r}}{(2r)!}x^r + \cdots\right]$$

and corresponding to $c = \frac{1}{2}$, we have a second solution,

$$y = a_0'\sqrt{x}\left[1 - \frac{2}{3}x + \frac{2}{15}x^2 - \frac{4}{315}x^3 + \cdots + (-1)^r\frac{2^{2r}}{(2r+1)!}x^r + \cdots\right].$$

It is easily shown that these two series converge for all values of x. Hence we have two distinct and valid solutions of our differential equation; their sum, with a_0 and a_0' arbitrary constants, is the general solution.

At times, not every part, perhaps none, of the general solution will be representable by series of the form (2), and consequently the general solution cannot be found as it was in the above example.

EXERCISES

1. Using the substitution $x = v^2$, transform the differential equation in the example of Art. 15 into an equation with constant coefficients, which may be solved by the method of Art. 11. Hence show that the general solution of the original equation is $y = c_1 \sin 2\sqrt{x} + c_2 \cos 2\sqrt{x}$, and verify the series solutions.

Apply the method of Frobenius to each of the differential equations in Exercises 2–6. Wherever possible, express the solution in finite form; otherwise, examine the series for convergence.

2. $(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0.$

3. $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (x^2+2)y = 0.$

4. $2x\frac{d^2y}{dx^2} + (1-4x)\frac{dy}{dx} - (1-2x)y = 0.$

5. $3x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0.$

6. $2x(1-x^2)^2\frac{d^2y}{dx^2} + (1-x^2)(1+3x^2)\frac{dy}{dx} + 3x(1+x^2)y = 0.$

7. Use the method of Frobenius to find the general solution of Bessel's equation of order n,

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0,$$

when n is not an integer. Show also that, when n is a positive integer or zero, Bessel's function of order n,

$$J_n(x) = \frac{x^n}{2^n n!} - \frac{x^{n+2}}{2^{n+2}1!(n+1)!} + \frac{x^{n+4}}{2^{n+4}2!(n+2)!} - \cdots + \frac{(-1)^r x^{n+2r}}{2^{n+2r}r!(n+r)!} + \cdots$$

is a particular solution. This series converges for all values of x.

8. Use the method of Frobenius to find the general solution of Legendre's equation,

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0.$$

Show also that, when n is a positive integer or zero, Legendre's polynomial,

$$P_n(x) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}\left[x^n - \frac{n(n-1)x^{n-2}}{2(2n-1)} + \frac{n(n-1)(n-2)(n-3)x^{n-4}}{2\cdot 4(2n-1)(2n-3)} - \cdots\right]$$

is a particular solution.

9. Show that the hypergeometric equation,

$$x(1-x)\frac{d^2y}{dx^2} + [\gamma - (\alpha+\beta+1)x]\frac{dy}{dx} - \alpha\beta y = 0,$$

where γ is not an integer or zero, has the hypergeometric series,

$$F(\alpha,\beta,\gamma,x) = 1 + \frac{\alpha\beta}{\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{3!\gamma(\gamma+1)(\gamma+2)}x^3 + \cdots,$$

convergent for $|x| < 1$, as a particular solution. Show also that $x^{1-\gamma}F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, x)$ is a solution.

10. Show that the method of Frobenius does not apply to the equation

$$x^4\frac{d^2y}{dx^2} + 2x^3\frac{dy}{dx} + y = 0.$$

Using a series of descending powers of x,

$$y = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_r}{x^r} + \cdots,$$

find the general solution of the above equation.

CHAPTER II

PARTIAL DIFFERENTIATION AND SPACE GEOMETRY

The formulation of a partial differential equation, to be considered in Chapter III, whether upon formal, geometric, or physical grounds, depends largely upon the techniques of partial differentiation and of solid analytic geometry. Likewise, the methods of solving differential equations and the interpretation of their solutions, discussed in later chapters, rest on these two types of techniques. Accordingly, this chapter is concerned with these preliminary basic matters. Nearly all of the topics discussed here may be found in any comprehensive calculus textbook, and consequently need be only briefly treated. Like the material of the preceding chapter, the present work may thus be used for review and reference purposes.

16. First partial derivatives. Let z be given as a function of two independent variables, x and y,

$$z = f(x, y). \tag{1}$$

If y is assigned some fixed value, or is thought of as fixed, z will vary only as a consequence of a change in x. When x takes on an increment Δx, z will change by an amount Δz such that

$$\Delta z = f(x + \Delta x, y) - f(x, y).$$

Then

$$\frac{\Delta z}{\Delta x} = \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}, \tag{2}$$

and if Δx is allowed to approach zero, the difference quotient (2) may approach a limit. When this limit exists, it is called the partial derivative of z with respect to x:

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}. \tag{3}$$

Similarly, if x is held fixed and y given an increment Δy, we are led, in general, to the limit

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}, \tag{4}$$

the partial derivative of z with respect to y.

Assuming that the partial derivatives (3) and (4) exist, we have the relations

$$\frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} - \frac{\partial f(x, y)}{\partial x} = \epsilon_1, \tag{5}$$

$$\frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} - \frac{\partial f(x, y)}{\partial y} = \epsilon_2, \tag{6}$$

where $\epsilon_1 \to 0$ as $\Delta x \to 0$ and $\epsilon_2 \to 0$ as $\Delta y \to 0$.

Other symbols for these two partial derivatives of the function (1) are

$$\frac{\partial f}{\partial x},\ f_x,\ p, \qquad \text{and} \qquad \frac{\partial f}{\partial y},\ f_y,\ q,$$

respectively. In all of the following work, the letters p and q will be reserved for use in this connection, and will be frequently employed.

The functional relation (1) may be represented geometrically by a surface in three-dimensional space, referred to three mutually perpendicular Cartesian coordinate axes x, y, z. Assigning to y a permissible constant value may also be regarded geometrically as cutting the surface (1) by a plane parallel to the xz-plane, thereby obtaining a curve of intersection. The slope of this curve at any point on it is then given by the value of $p = \partial z/\partial x$ at that point.

Similarly, if the surface (1) is cut by a plane parallel to the yz-plane, the slope of the curve of intersection at any point P on it is given by the value of $q = \partial z/\partial y$ at P.

Example 1. Find the slope of the curve of intersection of the surface $z = 2x^2y - 4xy^2$ with the plane $y = 2$ at the point $(-1, 2, 20)$.

Solution. Here we have

$$p = \frac{\partial z}{\partial x} = 4xy - 4y^2,$$

whence the value of p at the given point, namely, -24, is the slope desired.

If z is a function of three or more variables, say

$$z = F(x_1, x_2, \cdots, x_n), \tag{7}$$

we have n first partial derivatives:

$$\frac{\partial z}{\partial x_1} = \frac{\partial F}{\partial x_1} = p_1, \quad \frac{\partial z}{\partial x_2} = \frac{\partial F}{\partial x_2} = p_2, \cdots, \frac{\partial z}{\partial x_n} = \frac{\partial F}{\partial x_n} = p_n. \tag{8}$$

Each derivative p_j is found by differentiating (7) partially with respect to x_j, holding the $n - 1$ remaining x's constant.

When z is defined as a function of two or more independent variables by means of an implicit functional relation, rather than explicitly as in (1) or (7), term-by-term differentiation of the defining equation may be employed as usual. In such cases it is essential to keep clearly in mind which is the dependent and which are the independent variables.

Example 2. Given the cone $xy + yz + zx = 0$, find $\partial z/\partial y$.

Solution. In order that the symbol $\partial z/\partial y$ shall have meaning, z must be regarded as the dependent variable and x and y as the independent variables. Then term-by-term differentiation with respect to y, holding x fast and remembering that z is defined as a function of x and y by means of the given relation, gives us

$$x + y\frac{\partial z}{\partial y} + z + x\frac{\partial z}{\partial y} = 0,$$

$$\frac{\partial z}{\partial y} = -\frac{x + z}{x + y}.$$

In this case, the explicit functional relation could readily be found and differentiated with respect to y. The result would, of course, be identical with the expression obtained from the above value by replacing z by its equivalent in terms of x and y.

17. Higher partial derivatives. If $z = f(x, y)$, the derivatives p and q of z with respect to x and y, respectively, are themselves functions of x and y. These derived functions may in turn be differentiated partially, to obtain the four partial derivatives of second order,

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}, \qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}.$$

Each of these functions has, in turn, two partial derivatives, and so on.

If the two second-order "cross-derivatives" $\partial^2 z/\partial y \partial x$ and $\partial^2 z/\partial x \partial y$ are continuous functions of x and y, they are identical; thus, the order of differentiation is immaterial. Similar statements hold for cross-derivatives of higher orders and for functions of three or more independent variables. We shall assume throughout that the necessary conditions of continuity are fulfilled.

Other notations for the second partial derivatives of a function $f(x, y)$ of two independent variables x and y are

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = r, \qquad \frac{\partial^2 f}{\partial y^2} = f_{yy} = t,$$

$$\frac{\partial^2 f}{\partial y \partial x} = f_{xy} = s = f_{yx} = \frac{\partial^2 f}{\partial x \partial y}.$$

In conjunction with the symbols p and q for the first partial derivatives, we shall frequently use the notation r, s, t for the second partial derivatives.

Example. Find the second partial derivatives of the function $z = 2x^2y - 4xy^2$.

Solution. The first partial derivatives are

$$p = 4xy - 4y^2, \qquad q = 2x^2 - 8xy.$$

Hence we have

$$r = 4y, \qquad s = 4x - 8y, \qquad t = -8x.$$

EXERCISES

In Exercises 1–10, find the first and second partial derivatives of the given explicit functions.

1. $z = 3x^3y^2 + 4xy - 2y^3 - 5x + 2.$
2. $z = 4x^2e^{3y} - 5y^2e^{-x}.$
3. $z = \sqrt{x^2 - y^2} + \sin xy.$
4. $z = \sin(2x - 3y)\cos(3x - 2y).$
5. $z = \arctan xy - \log(1 + x^2y^2).$
6. $z = 4x \sec y + 3y \tan x.$
7. $w = 5x^2y^2z^2 - 3xyz + 1.$
8. $w = xy \sin z + yz \sin x + zx \sin y.$
9. $w = 2xye^z - 2ze^{xy}.$
10. $w = \dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}.$

In Exercises 11–15, find $\partial z/\partial x$ and $\partial z/\partial y$ using the given implicit functional relations.

11. $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx + 2x + 2y + 2z = 15.$
12. $xye^z + yze^x + zxe^y = 1.$
13. $8 \sin x \sin y \sin z = 1.$
14. $xy\sqrt{z} + yz\sqrt{x} + zx\sqrt{y} = 3.$
15. $(\log x)(\log y)(\log z) = 1.$
16. If $z = 4x^2 - 3xy + 5y^2$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2z.$

17. If $z = 3\frac{x}{y} - 7\frac{y}{x}$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$.

18. If $z = \log\sqrt{x^2 + y^2}$, show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

19. If $z = \sin(x - y) + e^{x+y}$, show that $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$.

20. If $z = \sqrt{2y - x} + \log(y + 3x)$, show that $2\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} - 3\frac{\partial^2 z}{\partial y^2} = 0$.

21. If $w = (x - y)(y - z)(z - x)$, show that $\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 0$.

22. If $w = \frac{x}{y + z} + \frac{y}{x + z} + \frac{z}{x + y}$, show that $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 0$.

23. The surface $x^2 + y^2 - 2z^2 = 0$ is cut by the plane $x = 1$. Find the angle at which the tangent line to the curve of intersection, at the point $(1, 7, 5)$, cuts the xy-plane. Identify the surface and curve, and draw a figure.

24. The surface $z = 2x^2 + 3y^2$ is cut by the plane $y = 2$. Find the angle at which the tangent line to the curve of intersection, at the point $(1, 2, 14)$, cuts the xy-plane. Identify the surface and curve, and draw a figure.

25. Find the equations of the tangent line to the curve $2x^2 + 4y^2 + 3z^2 = 34$, $x = 3$, at the point $(3, 1, 2)$. Identify the curve, and draw a figure.

18. Functions of functions. Let z be given as a function of two variables x and y,

$$z = f(x, y). \tag{1}$$

If x and y respectively take on increments Δx and Δy, z will be changed by a quantity Δz such that

$$z + \Delta z = f(x + \Delta x, y + \Delta y).$$

Hence, from (1), the increment Δz in z is

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y). \tag{2}$$

Subtracting and adding $f(x, y + \Delta y)$ in the right member, we have

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) + f(x, y + \Delta y) - f(x, y). \tag{3}$$

Now from relation (5) of Art. 16,

$$f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) = \frac{\partial f(x, y + \Delta y)}{\partial x}\Delta x + \epsilon_1 \Delta x. \tag{4}$$

Morever, since $\lim\limits_{\Delta y \to 0} \frac{\partial f(x, y + \Delta y)}{\partial x} = \frac{\partial f(x, y)}{\partial x}$, we have

$$\frac{\partial f(x, y + \Delta y)}{\partial x} = \frac{\partial f(x, y)}{\partial x} + \epsilon',$$

where $\epsilon' \to 0$ as $\Delta y \to 0$. Therefore

$$f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) = \frac{\partial f(x, y)}{\partial x} \Delta x + (\epsilon_1 + \epsilon')\Delta x,$$

where ϵ_1 and ϵ' approach zero with Δx and Δy. Likewise, we have from (6) of Art. 16,

$$f(x, y + \Delta y) - f(x, y) = \frac{\partial f(x, y)}{\partial y} \Delta y + \epsilon_2 \Delta y. \tag{5}$$

Combining (4) and (5) with (3), we then find

$$\Delta z = \frac{\partial f(x, y)}{\partial x} \Delta x + \frac{\partial f(x, y)}{\partial y} \Delta y + (\epsilon_1 + \epsilon') \Delta x + \epsilon_2 \Delta y, \tag{6}$$

where ϵ_1, ϵ', and ϵ_2 approach zero when Δx and Δy both approach zero. The first two terms in the right member of (6), containing the first partial derivatives of z, constitute what is known as the principal part of the increment Δz.

Suppose now that x and y are themselves functions of some variable u. By expressing x and y in (1) in terms of u, z becomes a function of a single variable u, and the derivative dz/du may be found in the usual manner. We may also, however, obtain this rate of change of z with respect to u as follows. Give to u an increment Δu, as a consequence of which x, y, and z respectively take on increments Δx, Δy, and Δz. Dividing (6), member for member, by Δu, we get

$$\frac{\Delta z}{\Delta u} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta u} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta u} + (\epsilon_1 + \epsilon')\frac{\Delta x}{\Delta u} + \epsilon_2 \frac{\Delta y}{\Delta u},$$

and if we now allow Δu to approach zero, whence Δx, Δy, ϵ_1, ϵ', and ϵ_2 all approach zero, we find

$$\frac{dz}{du} = \frac{\partial f}{\partial x}\frac{dx}{du} + \frac{\partial f}{\partial y}\frac{dy}{du}. \tag{7}$$

Thus we have the following

Theorem. *If z is a function of two variables x and y, and x and y are both functions of a single variable u, then the total derivative of z with respect to u is*

$$\frac{dz}{du} = \frac{\partial z}{\partial x}\frac{dx}{du} + \frac{\partial z}{\partial y}\frac{dy}{du}.$$

This theorem may be extended to the case in which z is a function of three or more variables, each of which is a function of u. We have

Corollary I. *If z is a function of n variables $x_1, x_2, \cdots, x_n$, and each x_j is a function of a single variable u, then the total derivative of z with respect to u is*

$$\frac{dz}{du} = \frac{\partial z}{\partial x_1}\frac{dx_1}{du} + \frac{\partial z}{\partial x_2}\frac{dx_2}{du} + \cdots + \frac{\partial z}{\partial x_n}\frac{dx_n}{du}.$$

A further extension may be made to the general case in which each of the variables in the functional expression for z is a function of two or more variables. All derivatives involved are then partial derivatives.

Corollary II. *If z is a function of n variables $x_1, x_2, \cdots, x_n$, and each x_j is a function of m variables $u_1, u_2, \cdots, u_m$, then the m first partial derivatives of z with respect to the u's are*

$$\frac{\partial z}{\partial u_1} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial u_1} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial u_1} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial u_1},$$

$$\frac{\partial z}{\partial u_2} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial u_2} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial u_2} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial u_2},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot,$$

$$\frac{\partial z}{\partial u_m} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial u_m} + \frac{\partial z}{\partial x_2}\frac{\partial x_2}{\partial u_m} + \cdots + \frac{\partial z}{\partial x_n}\frac{\partial x_n}{\partial u_m}.$$

Example. If $w = x^3y^2z^2 - 2xy^2$ and $x = u^2 + v^3$, $y = u^3 + v^2$, $z = 2uv$, find $\partial w/\partial u$ and $\partial w/\partial v$.

Solution. From Corollary II we have

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$= (3x^2y^2z^2 - 2y^2)\cdot 2u + (2x^3yz^2 - 4xy)\cdot 3u^2 + 2x^3y^2z\cdot 2v,$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

$$= (3x^2y^2z^2 - 2y^2)\cdot 3v^2 + (2x^3yz^2 - 4xy)\cdot 2v + 2x^3y^2z\cdot 2u.$$

19. Total differentials. If z is a function of two independent variables x and y,

$$z = f(x, y), \tag{1}$$

then the total differential dz of z is defined as the principal part of the increment Δz given by (6) of Art. 18; that is,

$$dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y. \tag{2}$$

The differentials of the independent variables x and y are respectively defined as the increments Δx and Δy, and consequently we may write, instead of (2),

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy. \tag{3}$$

The differential dz may be interpreted geometrically as the portion of the increment Δz, or of Δz produced, cut off by the plane tangent to the surface $z = f(x, y)$ at the point (x, y, z).

Now if x and y are not independent, but are functions of one or more other variables, say u and v, then by Corollary II of the theorem of Art. 18,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

Multiplying these equations respectively by $du = \Delta u$ and $dv = \Delta v$, we get

$$\frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv\right).$$

Since u and v are now the independent variables, definition (3) tells us that

$$dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv, \qquad dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv, \qquad dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv.$$

Hence in this case also,

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy.$$

This result may be extended to yield the

THEOREM. *If z is a function of n variables $x_1, x_2, \cdots, x_n$, the total differential of z is*

$$dz = \frac{\partial z}{\partial x_1} dx_1 + \frac{\partial z}{\partial x_2} dx_2 + \cdots + \frac{\partial z}{\partial x_n} dx_n.$$

This relation holds when the x's are functions of any number of other variables as well as when they are independent variables.

Example. If $P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz = 0$ is integrable, that is, if this differential relation is satisfied by some one-parameter family of surfaces $\phi(x, y, z) = c$, find the condition that the functions P, Q, R must fulfil.

Solution. From the functional relation $\phi(x, y, z) = c$, together with the above theorem, we get

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = 0.$$

In order that this shall lead to $P\,dx + Q\,dy + R\,dz = 0$, the partial derivatives of ϕ must be respectively proportional to the functions P, Q, R, or

$$\frac{\partial \phi}{\partial x} = \lambda P, \qquad \frac{\partial \phi}{\partial y} = \lambda Q, \qquad \frac{\partial \phi}{\partial z} = \lambda R.$$

Then

$$\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial}{\partial y}(\lambda P) = \lambda \frac{\partial P}{\partial y} + P \frac{\partial \lambda}{\partial y}$$

and

$$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial}{\partial x}(\lambda Q) = \lambda \frac{\partial Q}{\partial x} + Q \frac{\partial \lambda}{\partial x}.$$

But $\partial^2\phi/\partial y \partial x = \partial^2 \phi/\partial x \partial y$, and therefore

$$\lambda \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q \frac{\partial \lambda}{\partial x} - P \frac{\partial \lambda}{\partial y}. \tag{4}$$

Similarly, from the equality of the other two pairs of cross-derivatives, we find

$$\lambda \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) = R \frac{\partial \lambda}{\partial y} - Q \frac{\partial \lambda}{\partial z}, \tag{5}$$

$$\lambda \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = P \frac{\partial \lambda}{\partial z} - R \frac{\partial \lambda}{\partial x}. \tag{6}$$

Multiplying equations (4), (5), and (6) respectively by R, P, and Q, and adding, we get after cancellation,

$$P \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0. \tag{7}$$

This condition is therefore necessary for the integrability of $P\,dx + Q\,dy + R\,dz = 0$. It may also be shown that relation (7) is a sufficient condition.

20. Implicit functions. If y is given as an implicit function of x by means of an equation $f(x, y) = 0$, then, setting $z = f(x, y)$ temporarily, we get from the theorem of Art. 19,

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

Since $z = 0$ identically, $dz = 0$. This leads to

THEOREM I. *Let y be given as an implicit function of x by means of a relation $f(x, y) = 0$. Then the derivative of y with respect to x is*

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}, \quad \left(\frac{\partial f}{\partial y} \neq 0\right). \tag{1}$$

Suppose now that z is defined as a function of two independent variables x and y through an implicit relation $F(x, y, z) = 0$. Setting $w = F(x, y, z)$, we have

$$dw = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz = 0. \tag{2}$$

Also, since z is a function of x and y, we have

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy. \tag{3}$$

Inserting this expression for dz in (2), we get

$$\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}\right) dx + \left(\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y}\right) dy = 0. \tag{4}$$

Putting $dy = 0$, and taking $dx \neq 0$, (4) leads to

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0. \tag{5}$$

Likewise, setting $dx = 0$, $dy \neq 0$, (4) gives us

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0. \tag{6}$$

Relations (5) and (6) together yield

THEOREM II. *Let z be given as an implicit function of two independent variables x and y by means of an equation $F(x, y, z) = 0$. Then the first partial derivatives of z with respect to x and y are*

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}}, \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}, \qquad \left(\frac{\partial F}{\partial z} \neq 0\right). \tag{7}$$

Example. If $xy + yz + zx = 0$, find $\partial z/\partial y$ using Theorem II.

Solution. Here $F(x, y, z) = xy + yz + zx$, so that $F_y = x + z$ and $F_z = y + x$. Hence

$$\frac{\partial z}{\partial y} = -\frac{x + z}{x + y}.$$

This result agrees with that found in Example 2, Art. 16.

EXERCISES

In Exercises 1–4, find the total derivative of each function with respect to u.

1. $z = 4x^2 - 3xy + 5y^2,\ x = u^2 - 1,\ y = u^3 - u.$
2. $z = x \cos y + y \cos x,\ x = e^u,\ y = e^{-u}.$
3. $w = \log xyz,\ x = e^u,\ y = e^{2u},\ z = e^{3u}.$
4. $w = xye^z - ze^{xy},\ x = u,\ y = 1/u,\ z = \log u.$

In Exercises 5–8, find the first partial derivatives of each function with respect to u and v.

5. $z = x^3y + 2xy^3,\ x = 2u + v,\ y = u - 5v.$
6. $z = ye^{2x} - 3xe^{-y},\ x = \log (u + v),\ y = \log (u - v).$
7. $z = \sin xy - x \sin y,\ x = 4u^2 + 2v,\ y = 3v - 4v^2.$
8. $w = x^2y^2z^2 - xyz,\ x = e^{u+v},\ y = e^{u-v},\ z = e^{uv}.$

In Exercises 9–10, find dy/dx.

9. $x^3 + y^3 - 3axy = 0.$

10. $x \sin y + y \sin x = 1.$

In Exercises 11–12, find $\partial z/\partial x$ and $\partial z/\partial y$.

11. $(x + y) \sin z + z \sin (x - y) = 0.$

12. $(x^2 + y^2)e^z + ze^{xy} = 2.$

13. If $z = f(x, y)$ and $g(x, y) = 0$, find dz/dx.
14. If $z = f(y + ax) + g(y - ax)$, show that $r = a^2t$.
15. If $w = F(x - y, y - z, z - x)$, show that $w_x + w_y + w_z = 0$.
16. If $z = f(x, y)$, and $x = \rho \cos \theta$, $y = \rho \sin \theta$, show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

17. A homogeneous function $F(x_1, x_2, \cdots, x_n)$ is such that $F(\lambda x_1, \cdots, \lambda x_n) = \lambda^m F(x_1, \cdots, x_n)$ identically for any quantity λ. By differentiating with respect to λ, and then setting $\lambda = 1$ in the result, obtain Euler's theorem on homogeneous functions,

$$x_1 \frac{\partial F}{\partial x_1} + x_2 \frac{\partial F}{\partial x_2} + \cdots + x_n \frac{\partial F}{\partial x_n} = mF.$$

18. If $x = f(u, v)$ and $y = g(u, v)$, find $\partial u/\partial x$ and $\partial v/\partial y$ in terms of the partial derivatives of f and g with respect to u and v.

19. If u and v are functions of x and y such that $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$, and $x = \rho\cos\theta$, $y = \rho\sin\theta$, show that

$$\rho\frac{\partial u}{\partial \rho} = \frac{\partial v}{\partial \theta}, \qquad \frac{\partial u}{\partial \theta} = -\rho\frac{\partial v}{\partial \rho}, \qquad \rho^2\frac{\partial^2 u}{\partial \rho^2} + \rho\frac{\partial u}{\partial \rho} + \frac{\partial^2 u}{\partial \theta^2} = 0.$$

20. If u and v are given implicitly as functions of x and y by the relations $F(x, y, u, v) = 0$, $G(x, y, u, v) = 0$, find expressions for $\partial u/\partial y$ and $\partial v/\partial x$ in terms of first partial derivatives of F and G.

21. Tangent plane and normal line to a surface. It was stated in Art. 16 that when a surface $F(x, y, z) = 0$ is cut by a plane $y = y_0$, the slope of the curve of intersection, at any point P:(x_0, y_0, z_0) on it, is given by the value of $p = \partial z/\partial x$ at P. Likewise, the curve of intersection of $F(x, y, z) = 0$ with a plane $x = x_0$ has as slope the value of $q = \partial z/\partial y$ at P. These facts determine the equation of the plane tangent to the surface at P.

We suppose, for definiteness, that the tangent plane at P is not parallel to the z-axis, so that it may be written in the form

$$z - z_0 = A(x - x_0) + B(y - y_0). \tag{1}$$

Since the slope of the tangent line cut from the plane (1) by the plane $y = y_0$ is A, and the slope of the tangent line cut from (1) by $x = x_0$ is B, we have

$$A = p]_P, \qquad B = q]_P, \tag{2}$$

where the right members denote the values at P of the first partial derivatives of z obtained from the equation of the surface $F(x, y, z) = 0$. Hence (1) becomes

$$p]_P(x - x_0) + q]_P(y - y_0) - (z - z_0) = 0. \tag{3}$$

Using the expressions for p and q as given by Theorem II of Art. 20, we may write the equation of the tangent plane in the alternative form

$$F_x]_P(x - x_0) + F_y]_P(y - y_0) + F_z]_P(z - z_0) = 0, \tag{4}$$

where the coefficients represent the values at P of the three first partial derivatives of the function $F(x, y, z)$. This result holds whether or not the tangent plane is parallel to the z-axis. Accordingly, we have

THEOREM I. *The equation of the tangent plane to the surface $F(x, y, z) = 0$ at the point P:(x_0, y_0, z_0) on the surface may be written as*

$$p]_P(x - x_0) + q]_P(y - y_0) - (z - z_0) = 0,$$

whenever the tangent plane is not parallel to the z-axis, and in any event as

$$F_x]_P(x - x_0) + F_y]_P(y - y_0) + F_z]_P(z - z_0) = 0.$$

From analytic geometry, the direction cosines of a line perpendicular to the plane $Ax + By + Cz + D = 0$ are proportional to the coefficients A, B, C. Consequently the coefficients in the equation of the tangent plane are direction numbers (that is, numbers proportional to the direction cosines) of the normal line to the surface at P. Thus we also have

THEOREM II. *The equations of the normal line to the surface* $F(x, y, z) = 0$ *at the point* $P:(x_0, y_0, z_0)$ *on the surface may be written as*

$$\frac{x - x_0}{p]_P} = \frac{y - y_0}{q]_P} = \frac{z - z_0}{-1}$$

whenever the normal line is not perpendicular to the z-axis, and in any event as

$$\frac{x - x_0}{F_x]_P} = \frac{y - y_0}{F_y]_P} = \frac{z - z_0}{F_z]_P}.$$

The fact that F_x, F_y, and F_z (or, except as noted, p, q, and -1), evaluated at P, serve as direction numbers of the normal line to the surface $F(x, y, z) = 0$, will be of fundamental importance in our later work.

Example. Find the equation of the tangent plane and the equations of the normal line to the paraboloid $x + 2y^2 + 4z^2 - 4 = 0$ at: (*a*) the point $(-18, 3, 1)$; (*b*) the point $(2, 1, 0)$.

Solution. Setting $F(x, y, z) = x + 2y^2 + 4z^2 - 4$, we have

$$F_x = 1, \qquad F_y = 4y, \qquad F_z = 8z,$$

whence

$$p = -\frac{1}{8z}, \qquad q = -\frac{y}{2z}.$$

At the first point $(-18, 3, 1)$ we therefore get

$$p = -\tfrac{1}{8}, \qquad q = -\tfrac{3}{2}.$$

Thus the tangent plane at this point has the equation

$$-\tfrac{1}{8}(x + 18) - \tfrac{3}{2}(y - 3) - (z - 1) = 0,$$

or

$$x + 12y + 8z - 26 = 0,$$

and the normal line here has the equations

$$\frac{x+18}{1} = \frac{y-3}{12} = \frac{z-1}{8}.$$

At the second point (2, 1, 0), however, p and q fail to exist, and consequently we must use

$$F_x = 1, \qquad F_y = 4, \qquad F_z = 0.$$

The tangent plane at (2, 1, 0) is therefore

$$1(x-2) + 4(y-1) + 0(z-0) = 0,$$

or

$$x + 4y = 6,$$

and the normal line there is given by

$$\frac{x-2}{1} = \frac{y-1}{4}, \quad z = 0.$$

It is common practice to use the symmetrical form of the equations of a line even when one or two of the direction numbers are zero. Thus, the latter normal line equations are conventionally written

$$\frac{x-2}{1} = \frac{y-1}{4} = \frac{z}{0},$$

but it should be kept in mind that we mean by this symbolism merely the preceding pair of equations.

22. Angle between two lines; parallel and perpendicular lines. From analytic geometry, the angle θ, between two lines with respective direction numbers a_1, b_1, c_1 and a_2, b_2, c_2, is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}. \tag{1}$$

Since the angle between two planes is equal to the angle between their normals, formula (1) serves also to determine the angle between two planes whose normals have the stated direction numbers.

Let $F(x, y, z) = 0$ and $G(x, y, z) = 0$ be two surfaces intersecting at a point P. By the angle between these surfaces at P is meant the angle between their tangent planes at P. From the results obtained in Art. 21, together with formula (1), we get for the angle θ between the surfaces,

$$\cos\theta = \frac{F_xG_x + F_yG_y + F_zG_z}{\sqrt{F_x^2 + F_y^2 + F_z^2}\sqrt{G_x^2 + G_y^2 + G_z^2}}, \tag{2}$$

where the six partial derivatives concerned are supposed evaluated at P.

If two lines are parallel and similarly directed, their respective direction cosines are equal, and therefore two corresponding sets of direction numbers a_1, b_1, c_1 and a_2, b_2, c_2 are proportional:

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}. \tag{3}$$

Conversely, if (3) holds, the lines must be parallel.

If two lines are perpendicular, that is, if the angle θ between them is 90°, formula (1) yields the relation

$$a_1a_2 + b_1b_2 + c_1c_2 = 0. \tag{4}$$

Conversely, if (4) holds, the lines must be perpendicular.

Conditions (3) and (4), for parallelism and perpendicularity respectively, are of frequent utility.

23. Intersecting surfaces; space curves. Let two surfaces $F(x, y, z) = 0$ and $G(x, y, z) = 0$ intersect in a curve. Then the equations

$$F(x, y, z) = 0, \qquad G(x, y, z) = 0, \tag{1}$$

considered simultaneously, represent analytically that curve of intersection.

The analytic representation of a space curve is, of course, not unique. Thus, we may eliminate z and y in turn from equations (1), yielding the equations of two of the projecting cylinders of the curve:

$$y = f(x), \qquad z = g(x); \tag{2}$$

these two equations together likewise represent the curve.

Another method frequently used to represent a space curve is to express each coordinate, x, y, z, of the curve in terms of a parameter θ:

$$x = \phi_1(\theta), \qquad y = \phi_2(\theta), \qquad z = \phi_3(\theta). \tag{3}$$

If the parameter θ is eliminated between the first and second, and between the first and third, of equations (3), relations of the form (2) are obtained.

Let P:(x_0, y_0, z_0) and Q:$(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z)$ be two neighboring points on a space curve. The direction cosines of the secant PQ are then

$$\cos\alpha' = \frac{\Delta x}{PQ}, \qquad \cos\beta' = \frac{\Delta y}{PQ}, \qquad \cos\gamma' = \frac{\Delta z}{PQ}.$$

Let P be kept fixed and Q be allowed to move along the curve toward P, that is, let Δx, Δy, and Δz all approach zero. Then the secant PQ will approach, as its limiting position, the tangent line to the curve at P. If Δs is the length of arc from P to Q, the direction angles of the tangent line at P will be given by

$$\begin{aligned} \cos\alpha &= \lim \frac{\Delta x}{PQ} = \lim\left(\frac{\Delta x}{\Delta s}\cdot\frac{\Delta s}{PQ}\right) = \frac{dx}{ds}, \\ \cos\beta &= \lim \frac{\Delta y}{PQ} = \lim\left(\frac{\Delta y}{\Delta s}\cdot\frac{\Delta s}{PQ}\right) = \frac{dy}{ds}, \\ \cos\gamma &= \lim \frac{\Delta z}{PQ} = \lim\left(\frac{\Delta z}{\Delta s}\cdot\frac{\Delta s}{PQ}\right) = \frac{dz}{ds}, \end{aligned} \tag{4}$$

where each derivative is supposed evaluated at P. Thus the direction cosines are respectively equal to the rates of change, with respect to arc length, of the coordinates x, y, z. Furthermore, the differentials dx, dy, dz serve as direction numbers of the tangent line.

Example. Find the direction angles of the tangent line to the curve $x = 3\theta$, $y = 2\theta^2$, $z = 1 - \theta$ at the point (3, 2, 0).

Solution. Since $dx = 3\,d\theta$, $dy = 4\theta\,d\theta$, and $dz = -\,d\theta$, a set of direction numbers of the tangent line at any point of parameter θ is 3, 4θ, -1. The given point corresponds to $\theta = 1$, and consequently the direction numbers here are 3, 4, -1. Hence the direction cosines are

$$\cos\alpha = \frac{3}{\sqrt{3^2 + 4^2 + (-1)^2}} = \frac{3}{\sqrt{26}}, \qquad \cos\beta = \frac{4}{\sqrt{26}}, \qquad \cos\gamma = -\frac{1}{\sqrt{26}},$$

whence we find

$$\alpha = 53^\circ\,58', \qquad \beta = 38^\circ\,20', \qquad \gamma = 101^\circ\,19'.$$

In rectangular coordinates, the planes $x =$ const., $y =$ const., and $z =$ const. are three mutually orthogonal surfaces and the element of volume $\Delta x\,\Delta y\,\Delta z$ is a rectangular parallelepiped, as shown in Fig. 1. From equations (4) and the relation $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, we also find that the differential of arc length is given by

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2. \tag{5}$$

In cylindrical coordinates (ρ, θ, z), we use the polar coordinates (ρ, θ) of two dimensions, augmented by the third (rectangular) variable z. The relations between rectangular and cylindrical coordinates are

$$x = \rho\cos\theta, \qquad y = \rho\sin\theta, \qquad z = z. \tag{6}$$

The mutually orthogonal families of surfaces $\rho =$ const., $\theta =$ const., and $z =$ const. are respectively circular cylinders, $x^2 + y^2 = c_1$, planes through the z-axis, $y/x = c_2$, and planes parallel to the xy-plane, $z = c_3$,

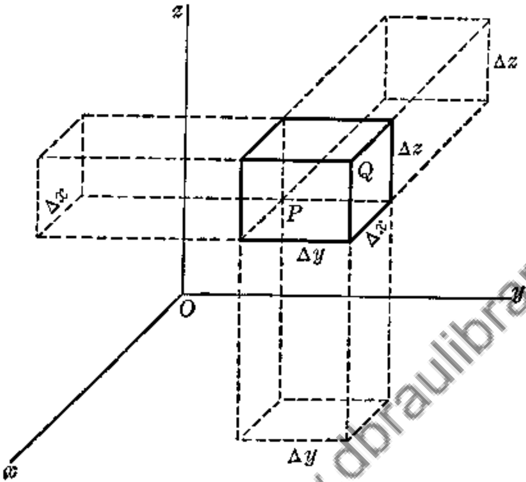

FIG. 1

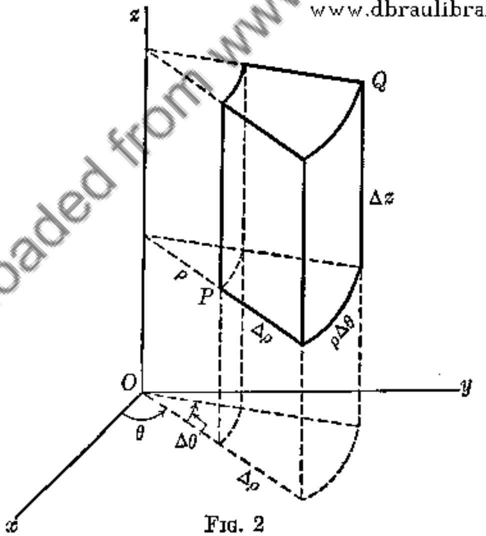

FIG. 2

and the element of volume, approximately equal to $\rho\, \Delta\rho\, \Delta\theta\, \Delta z$, is that shown in Fig. 2. By differentiating equations (6) and substituting in (5), we get

$$(ds)^2 = (d\rho)^2 + (\rho\, d\theta)^2 + (dz)^2. \tag{7}$$

For spherical coordinates (ρ, ϕ, θ), we use the distance ρ from the origin, the angle ϕ measured from the positive z-axis, and the angle θ from the xz-plane, so that

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho\cos\phi. \tag{8}$$

The mutually orthogonal families of surfaces $\rho =$ const., $\phi =$ const., and $\theta =$ const. are respectively spheres, $x^2 + y^2 + z^2 = c_1$, cones, $(x^2 + y^2)/z^2 = c_2$, and planes through the z-axis, $y/x = c_3$, and the element of volume, approximately equal to $\rho^2 \sin\phi\, \Delta\rho\, \Delta\phi\, \Delta\theta$, is that shown in Fig. 3. Differentiation of equations (8) and substitution in (5) yields

$$(ds)^2 = (d\rho)^2 + (\rho\, d\phi)^2 + (\rho \sin\phi\, d\theta)^2. \tag{9}$$

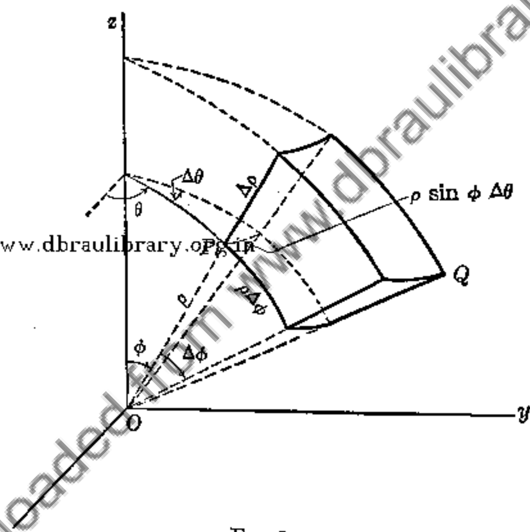

FIG. 3

The above discussion of particular coordinate systems may be extended to the general system of curvilinear coordinates employing any triply orthogonal families of surfaces, $u(x, y, z) = c_1$, $v(x, y, z) = c_2$, $w(x, y, z) = c_3$. The surfaces $u = c_1$, $u + \Delta u = c_1$, $v = c_2$, $v + \Delta v = c_2$, $w = c_3$, and $w + \Delta w = c_3$ will then bound an element with edges $E_1\,\Delta u$, $E_2\,\Delta v$, $E_3\,\Delta w$ and of approximate volume $E_1E_2E_3\,\Delta u\,\Delta v\,\Delta w$, where $E_1(u, v, w)$, $E_2(u, v, w)$, and $E_3(u, v, w)$ are three functions of position. Furthermore, it turns out that the differential of arc length is then given by

$$(ds)^2 = (E_1\, du)^2 + (E_2\, dv)^2 + (E_3\, dw)^2. \tag{10}$$

We shall make use of orthogonal curvilinear coordinates in the next chapter in connection with a problem in heat flow. This general procedure will then enable us easily to express our functional relations in terms of rectangular, cylindrical, or spherical coordinates, whichever is most useful for a particular purpose.

EXERCISES

In Exercises 1–6, find the equation of the tangent plane and the equations of the normal line to the given surfaces at the points indicated.

1. $x^2 + y^2 + z^2 = 21$; $(2, -1, 4)$.
2. $3x^2 + 2y^2 = 35$; $(-3, 2, 5)$.
3. $x^2 - 3y^2 + 2z^2 = 6$; $(4, 2, -1)$.
4. $2x^2 - y^2 - 4z^2 = 8$; $(2, 0, 0)$.
5. $x^2 - 2y^2 + z^2 = 0$; $(-1, -1, 1)$.
6. $xyz = 6$; $(-2, 3, -1)$.

7. Find the angle between the ellipsoid $2x^2 + 4y^2 + z^2 = 16$ and the plane $x - 3y + 4z = 7$ at the point $(2, 1, 2)$.

8. Show that the paraboloid $x^2 + 2y^2 - 3z = 6$ and the hyperboloid $2x^2 + 4y^2 - 3z^2 = 15$ are tangent to each other at the point $(-1, 2, 1)$.

9. Show that the ellipsoid $5x^2 + 9y^2 + 45z^2 = 45$ and the hyperboloid $3x^2 + 7y^2 - 21z^2 = 21$ cut orthogonally at each point of intersection.

10. Find the values of a and b in order that the cone $x^2 = ay^2 + bz^2$ shall intersect the hyperboloid $2x^2 - y^2 - 3z^2 + 20 = 0$ orthogonally at the point $(2, 1, 3)$.

11. Find the equations of the normal to the paraboloid $x^2 + 2z^2 = 8y + 2$ which is parallel to the line $3 - x = y - 5 = 2z - 6$.

12. Find the angle at which the normal line to the surface $xyz - xy - yz - xz = 8$ at the point $(2, -1, -2)$ cuts the sphere $x^2 + y^2 + z^2 = 3$.

13. Show that each sphere of the family $x^2 + y^2 + z^2 + c_1x = 0$ cuts every sphere of the family $x^2 + y^2 + z^2 + c_2y = 0$ orthogonally.

14. Show that the sum of the intercepts on the coordinate axes of the tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}$ is independent of the position of the point of tangency.

15. Show that the sum of the squares of the intercepts on the coordinate axes of the tangent plane to the surface $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$ is constant.

16. Show that the volume of the tetrahedron formed by the coordinate planes and the tangent plane to the surface $xyz = a^3$ is constant.

17. Find the direction angles of the tangent line to the helix $x = \cos\theta$, $y = \sin\theta$, $z = \theta$ at the point $(-1, 0, \pi)$.

18. Find the direction angles of the ellipse $x^2 + 2y^2 + 4z^2 = 10$, $x - 2y = 0$, at the point $(2, 1, -1)$.

19. Find the angle at which the curve $x^2 + y^2 + z^2 = 4$, $x^2 + y^2 = 2x$ cuts the surface $y^2 + z^2 = 3x$.

20. Two space curves, having a point in common, are respectively represented by $x = f(\theta)$, $y = g(\theta)$, $z = h(\theta)$ and $x = F(\theta)$, $y = G(\theta)$, $z = H(\theta)$. Find the conditions that these curves (a) cut orthogonally; (b) are tangent.

24. Functional dependence; Jacobians.

Let $u = f(x, y)$ and $v = g(x, y)$ be two functions of the two variables x and y, and suppose that there exists a functional relation $\phi(u, v) = 0$ between u and v. By partial differentiation of the equation $\phi(u, v) = 0$ with respect to x and y in turn, we have

$$\begin{aligned} \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} &= 0, \\ \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} &= 0. \end{aligned} \tag{1}$$

These relations represent a system of two homogeneous linear algebraic equations in the quantities ϕ_u and ϕ_v, which cannot be identically zero since ϕ actually involves both u and v. From algebra, it is known that linear homogeneous equations can have solutions other than the trivial set of zero solutions only if the determinant of the system vanishes. Hence we must have

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\[2ex] \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix} = 0. \tag{2}$$

The determinant in (2), formed of the first partial derivatives of u and v as indicated, is called the Jacobian, or functional determinant, of u and v with respect to x and y. It is denoted briefly by the letter J when no ambiguity as to the variables involved can arise, or by the symbols

$$\frac{\partial(u, v)}{\partial(x, y)}, \qquad \frac{D(u, v)}{D(x, y)}. \tag{3}$$

The argument above, which can be readily extended to the case of n functions of n variables, shows that the vanishing of the Jacobian is necessary to the existence of functional dependence among the functions. It may also be shown that the vanishing of J is sufficient.* The general theorem may be stated as follows.

THEOREM I. *Let $u_1, u_2, \cdots, u_n$ be n functions of the n independent variables $x_1, x_2, \cdots, x_n$. These n functions will be functionally dependent, that is, there will exist a relation $\phi(u_1, u_2, \cdots, u_n) = 0$ which does not involve explicitly any of the variables $x_1, \cdots, x_n$, if and only if the Jacobian*

$$J = \frac{\partial(u_1, u_2, \cdots, u_n)}{\partial(x_1, x_2, \cdots, x_n)} \tag{4}$$

vanishes identically.

* See, for example, Goursat-Hedrick, "Mathematical Analysis," vol. I, page 53.

Example 1. Show that the functions

$$\arctan y^2 + 2 \arctan x \quad \text{and} \quad \frac{2x + y^2 - x^2y^2}{1 - x^2 - 2xy^2}$$

are dependent.

Solution. Denoting these functions by u and v respectively, we get

$$\frac{\partial u}{\partial x} = \frac{2}{1 + x^2}, \qquad \frac{\partial u}{\partial y} = \frac{2y}{1 + y^4},$$

$$\frac{\partial v}{\partial x} = \frac{2(1 + x^2)(1 + y^4)}{(1 - x^2 - 2xy^2)^2}, \qquad \frac{\partial v}{\partial y} = \frac{2y(1 + x^2)^2}{(1 - x^2 - 2xy^2)^2}.$$

Hence

$$J = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

$$= \frac{4y(1 + x^2)}{(1 - x^2 - 2xy^2)^2} - \frac{4y(1 + x^2)}{(1 - x^2 - 2xy^2)^2} = 0,$$

and therefore u and v are functionally dependent. In this case, the functional relation between u and v (which cannot, in general, be readily determined) is $\tan u - v = 0$.

The Jacobian is also of importance in connection with implicit functions. We merely state, without proof, the following theorem; this is a special case of a more general theorem which we need not consider here.

THEOREM II. *Let there be given two functional relations, $F(x, y, u, v) = 0$ and $G(x, y, u, v) = 0$, involving the two pairs of variables x, y and u, v. Let these equations be satisfied for the set of values $x = x_0$, $y = y_0$, $u = u_0$, $v = v_0$, and let the functions F and G and their first partial derivatives be continuous in the neighborhood of this set of values. Suppose further that the Jacobian*

$$J \equiv \begin{vmatrix} F_x & G_x \\ F_y & G_y \end{vmatrix} \tag{5}$$

is different from zero for this set of values. Then there exists one and only one system of continuous functions $x = \phi(u, v)$ and $y = \psi(u, v)$ satisfying the equations $F = 0$ and $G = 0$ identically and such that $x_0 = \phi(u_0, v_0)$, $y_0 = \psi(u_0, v_0)$.

Example 2. Consider the functional relations

$$F \equiv xv + x + 2yv + 2y - 5 = 0, \qquad G \equiv 4xu + 3yu - 5 = 0.$$

Here we have

$$F_x = v + 1, \qquad F_y = 2v + 2, \qquad G_x = 4u, \qquad G_y = 3u,$$

whence

$$J = 3u(v+1) - 8u(v+1) = -5u(v+1).$$

Thus $J \neq 0$ except for $u = 0$ or for $v = -1$. We should therefore expect that the given functional relations yield x and y as functions of u and v valid for all values of u and v except possibly $u = 0$ and $v = -1$. In this case we may easily obtain, from the given equations,

$$x = \frac{2}{u} - \frac{3}{v+1}, \qquad y = \frac{4}{v+1} - \frac{1}{u}.$$

These expressions evidently verify the prediction of the Jacobian.

If, in particular, the relations $F = 0$ and $G = 0$ are of the forms $u - f(x, y) = 0$, $v - g(x, y) = 0$, that is, if u and v are given explicitly as functions of x and y, Theorem II states the conditions under which the inverse functions $x = \phi(u, v)$, $y = \psi(u, v)$ exist. The Jacobian in this special case becomes

$$J = \begin{vmatrix} -f_x & -g_x \\ -f_y & -g_y \end{vmatrix} = f_x g_y - f_y g_x. \tag{6}$$

25. Envelopes of curves. Consider a one-parameter family of curves,

$$f(x, y, \alpha) = 0. \tag{1}$$

If, for each permissible value of the parameter α, equation (1) yields a curve tangent to one and the same curve C, then C is called the envelope of the family (1).

Let α_0 and $\alpha_0 + \Delta\alpha$ be two values of α differing but little. If the family (1) has an envelope C, the two curves

$$f(x, y, \alpha_0) = 0, \qquad f(x, y, \alpha_0 + \Delta\alpha) = 0 \tag{2}$$

will, in general, intersect near the points of tangency of these curves with C. Now the equation

$$\frac{f(x, y, \alpha_0) - f(x, y, \alpha_0 + \Delta\alpha)}{\Delta\alpha} = 0$$

represents another curve, belonging to the family (1), which passes through the point of intersection of the curves (2). Hence we may take, instead of the two curves (2), the curves

$$f(x, y, \alpha_0) = 0, \qquad \frac{f(x, y, \alpha_0) - f(x, y, \alpha_0 + \Delta\alpha)}{\Delta\alpha} = 0. \tag{3}$$

If we allow $\Delta\alpha$ to approach zero, the second of the curves (3) will approach the first, and their point of intersection will approach the point of tangency of the first curve with the envelope C. But

$$-\lim_{\Delta\alpha \to 0} \frac{f(x, y, \alpha_0) - f(x, y, \alpha_0 + \Delta\alpha)}{\Delta\alpha} = \frac{\partial f(x, y, \alpha_0)}{\partial\alpha}, \tag{4}$$

where the right member denotes the value, for $\alpha = \alpha_0$, of the partial derivative of $f(x, y, \alpha)$ with respect to α. Hence we have

THEOREM I. *If the family of curves $f(x, y, \alpha) = 0$ has an envelope, the envelope may, in general, be found by eliminating the parameter α from the equations*

$$f(x, y, \alpha) = 0, \qquad \frac{\partial f(x, y, \alpha)}{\partial\alpha} = 0.$$

Example. Find the envelope of the family of circles $x^2 + y^2 + 2\alpha x + 2\alpha y + \alpha^2 = 0$.

Solution. Setting $f(x, y, \alpha) = x^2 + y^2 + 2\alpha x + 2\alpha y + \alpha^2 = 0$, we have

$$\frac{\partial f}{\partial\alpha} = 2x + 2y + 2\alpha = 0,$$

so that

$$\alpha = -(x + y).$$

Replacing α by $-(x + y)$ in $f(x, y, \alpha) = 0$, we get as the eliminant $xy = 0$, or $x = 0$ and $y = 0$. Evidently these two equations constitute the envelope, for they represent the coordinate axes to which all the circles of the original family are tangent (Fig. 4).

The phrase "in general" was inserted in the statement of Theorem I because the equation $f(x, y, \alpha) = 0$ of the curve family may sometimes be given in such form that the process does not yield the envelope. Thus, if the equation of the circles of the above example is solved for α to yield $\alpha = -(x + y) \pm \sqrt{2xy}$, and we set $f(x, y, \alpha) = \alpha + x + y \mp \sqrt{2xy} = 0$, the derived relation $\partial f/\partial\alpha = 0$ leads to the absurdity $1 = 0$, so that the process breaks down. It is therefore essential that the equation of the family be taken in the proper form. To insure the validity of the process, we have the following sufficiency condition.

THEOREM II. *Let $f(x, y, \alpha)$ and its first and second partial derivatives be continuous in the neighborhood of the set of values $x = x_0$, $y = y_0$, $\alpha = \alpha_0$, for which $f(x, y, \alpha)$ and $\partial f/\partial \alpha$ vanish. If $\partial^2 f/\partial \alpha^2 \neq 0$ and the Jacobian J of f and f_α is likewise different from zero for that set of values, that is, if*

$$J \equiv \begin{vmatrix} f_x & f_y \\ f_{x\alpha} & f_{y\alpha} \end{vmatrix} \neq 0, \qquad f_{\alpha\alpha} \neq 0,$$

for $x = x_0$, $y = y_0$, $\alpha = \alpha_0$, then the equations

$$f(x, y, \alpha) = 0, \qquad \frac{\partial f(x, y, \alpha)}{\partial \alpha} = 0$$

define a curve to which each member of the family $f(x, y, \alpha) = 0$ in the neighborhood of (x_0, y_0, α_0) is tangent.

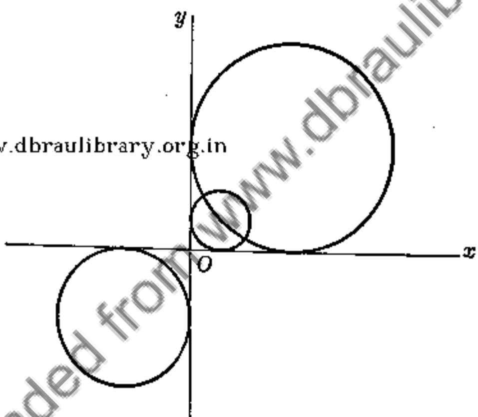

FIG. 4

This theorem may be proved with the aid of Theorem II of Art. 24. For, since $J \neq 0$, the equations $f(x, y, \alpha) = 0$, $f_\alpha(x, y, \alpha) = 0$ yield x and y as continuous functions of α,

$$x = \phi(\alpha), \qquad y = \psi(\alpha), \tag{5}$$

such that $x_0 = \phi(\alpha_0)$, $y_0 = \psi(\alpha_0)$. Now differentiation with respect to α of $f = 0$ and $f_\alpha = 0$, together with (5), gives us

$$f_x\phi' + f_y\psi' + f_\alpha = 0, \tag{6}$$

$$f_{x\alpha}\phi' + f_{y\alpha}\psi' + f_{\alpha\alpha} = 0. \tag{7}$$

Since $f_{\alpha\alpha} \neq 0$ by hypothesis, (7) shows that ϕ' and ψ' cannot both vanish. Also, since $f_\alpha = 0$, (6) becomes $f_x\phi' + f_y\psi' = 0$, whence equations (5) define a curve whose slope is

$$\frac{dy}{dx} = \frac{\psi'}{\phi'} = -\frac{f_x}{f_y}.$$

But f_x and f_y cannot both vanish since $J \neq 0$, and consequently the equation $f(x, y, \alpha) = 0$ defines a curve whose slope is

$$\frac{dy}{dx} = -\frac{f_x}{f_y}.$$

Therefore the curve (5) and the curve $f(x, y, \alpha) = 0$ are tangent for any value of α in the neighborhood of $\alpha = \alpha_0$, that is, equations (5) represent, in parametric form, the envelope of the curves $f(x, y, \alpha) = 0$.

When the circles of the above example are written in the form $\alpha + x + y \mp \sqrt{2xy} = 0$, neither of the conditions $J \neq 0$ and $f_{\alpha\alpha} \neq 0$ of Theorem II is fulfilled, and consequently the method is not applicable.

The process described in Theorem I may sometimes yield, in addition to or instead of an envelope, certain extraneous loci. Theorem II will then be of help in discarding all but the desired envelope.

26. Envelopes of surfaces. The preceding remarks apply with only small changes to the problem of finding the envelope of a family of surfaces. If there exists a surface S to which each member of a given one-parameter family of surfaces is tangent along a corresponding curve C, then S is said to be the envelope of the family, and each curve C of tangency is called a characteristic curve. We state, without further discussion, the following

THEOREM I. *If $F(x, y, z, \alpha) = 0$ represents a one-parameter family of surfaces possessing an envelope, that envelope may, in general, be found by eliminating the parameter α between the equations*

$$F(x, y, z, \alpha) = 0, \qquad \frac{\partial F(x, y, z, \alpha)}{\partial \alpha} = 0. \tag{1}$$

For each fixed value of α, equations (1) together define the corresponding characteristic curve. Thus the envelope is generated by the characteristic curves as α varies.

Example 1. Show that the family of planes $2\alpha x + (\alpha^2 - 1)y - (\alpha^2 + 1)z = 0$ possesses an envelope, and find its equation.

Solution. Setting $F(x, y, z, \alpha) = 2\alpha x + (\alpha^2 - 1)y - (\alpha^2 + 1)z = 0$, we have

$$\frac{\partial F}{\partial \alpha} = 2x + 2\alpha y - 2\alpha z = 0,$$

from which we find

$$\alpha = \frac{x}{z - y}.$$

Inserting this expression for α in $F(x, y, z, \alpha) = 0$, we get, after simplification,

$$x^2 + y^2 - z^2 = 0.$$

This cone is the envelope of the given planes, that is, the equation $F(x,y,z,\alpha) = 0$ represents the family of tangent planes OAB to the above cone (Fig. 5). For each value of α, $F = 0$ and $F_\alpha = 0$ together define an element OP of the cone, and this line is the corresponding characteristic curve.

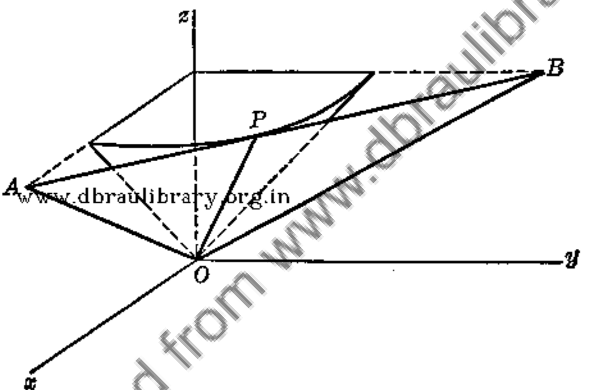

FIG. 5

The envelope of a one-parameter family of planes is called a developable surface. The characteristic curves are in this case straight lines, and consequently a developable surface is always a ruled surface. For, if $F(x, y, z, \alpha) = 0$ is linear in x, y, and z, so is $F_\alpha = 0$, and these two planes define a straight line for every α. Evidently the cone of Example 1 is a developable and ruled surface.

A two-parameter family of surfaces may also possess an envelope. In this connection, we have

THEOREM II. *If $G(x, y, z, \alpha, \beta) = 0$ represents a two-parameter family of surfaces possessing an envelope, that envelope may, in general, be found by eliminating the parameters α and β from the equations*

$$G(x, y, z, \alpha, \beta) = 0, \qquad \frac{\partial G}{\partial \alpha} = 0, \qquad \frac{\partial G}{\partial \beta} = 0.$$

Example 2. Show that the family of planes $2\alpha x + 2\beta y + z + \alpha^2 + \beta^2 = 0$ possesses an envelope, and find its equation.

Solution. Setting $G(x, y, z, \alpha, \beta) = 2\alpha x + 2\beta y + z + \alpha^2 + \beta^2 = 0$, we have

$$\frac{\partial G}{\partial \alpha} = 2x + 2\alpha = 0, \qquad \frac{\partial G}{\partial \beta} = 2y + 2\beta = 0,$$

whence $\alpha = -x, \beta = -y$. Elimination of α and β then gives us

$$-2x^2 - 2y^2 + z + x^2 + y^2 = 0,$$

or

$$z = x^2 + y^2.$$

Thus the given planes have a paraboloid of revolution as envelope (Fig. 6). In fact, it is easily shown by the method described in Art. 21 that the equation

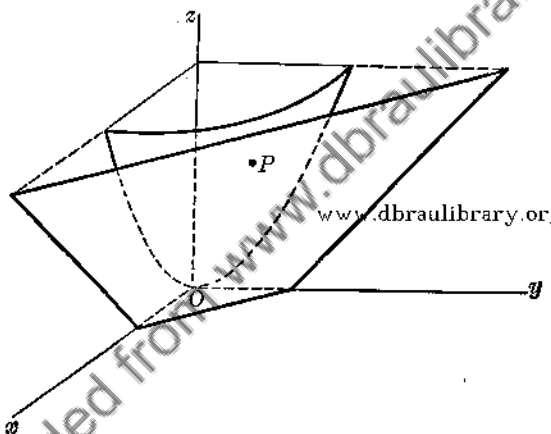

FIG. 6

$G = 0$ represents a plane tangent to the above paraboloid at the point $P:(-\alpha, -\beta, \alpha^2 + \beta^2)$.

It is easy to see that the family of all tangent planes to a given surface constitutes a two-parameter family of which the surface is the envelope.

EXERCISES

1. If $z = f(x, y)$ and $g(x, y) = 0$, show by means of Theorem I of Art. 24 that $dz/dx = 0$ identically if and only if $f(x, y)$ and $g(x, y)$ are functionally dependent. (Cf. Exercise 13, Art. 20.)

2. If $x = f(u, v)$ and $y = g(u, v)$, show that the first partial derivatives of u and v with respect to x and y do not exist if $f(u, v)$ and $g(u, v)$ are functionally dependent. (Cf. Exercise 18, Art. 20.)

3. If $F(x, y, u, v) = 0$ and $G(x, y, u, v) = 0$, show by means of Theorem II of Art. 24 that the first partial derivatives of u and v with respect to x and y do not exist if F and G as functions of u and v are functionally dependent. (Cf. Exercise 20, Art. 20.)

4. Generalize the result of Exercise 3 to the case of three functions of x, y, u, v, and w.

5. If $u_1 = f(y_1, y_2)$, $u_2 = g(y_1, y_2)$, and $y_1 = F(x_1, x_2)$, $y_2 = G(x_1, x_2)$, show that

$$\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = \frac{\partial(u_1, u_2)}{\partial(y_1, y_2)} \cdot \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}.$$

6. If $u = f(x, y)$ and $v = g(x, y)$ represent a transformation with non-vanishing Jacobian

$$J = \frac{\partial(u, v)}{\partial(x, y)},$$

and if the inverse transformation $x = F(u, v)$, $y = G(u, v)$ has the Jacobian

$$j = \frac{\partial(x, y)}{\partial(u, v)},$$

shows by means of the result of Exercise 5 that $jJ = 1$.

7. If $x = f(u, v)$, $y = g(u, v)$, $z = h(u, v)$, and x, y, z are functionally connected, show that

$$\frac{\partial z}{\partial x} = \frac{\dfrac{\partial(z, y)}{\partial(u, v)}}{\dfrac{\partial(x, y)}{\partial(u, v)}}, \qquad \frac{\partial z}{\partial y} = \frac{\dfrac{\partial(x, z)}{\partial(u, v)}}{\dfrac{\partial(x, y)}{\partial(u, v)}}, \qquad \left(\frac{\partial(x, y)}{\partial(u, v)} \neq 0\right).$$

8. If $F(x, y, u, v) = 0$ and $G(x, y, u, v) = 0$, show that

$$\frac{\partial(u, v)}{\partial(x, y)} = -\frac{\dfrac{\partial(F, G)}{\partial(x, y)}}{\dfrac{\partial(F, G)}{\partial(u, v)}}, \qquad \left(\frac{\partial(F, G)}{\partial(u, v)} \neq 0\right).$$

9. If the surfaces $F(x, y, z) = 0$ and $G(x, y, z) = 0$ intersect along a curve through the point $P:(x_0, y_0, z_0)$, show that the equations of the tangent line to the curve may be written in the form

$$\frac{x - x_0}{\left.\dfrac{\partial(F, G)}{\partial(y, z)}\right]_P} = \frac{y - y_0}{\left.\dfrac{\partial(F, G)}{\partial(z, x)}\right]_P} = \frac{z - z_0}{\left.\dfrac{\partial(F, G)}{\partial(x, y)}\right]_P}.$$

10. If the function $F(x, y)$ is transformed into $G(u, v)$ under the substitution $x = f(u, v)$, $y = g(u, v)$, it may be shown that the iterated integral of $F(x, y)$ over a region A of the xy-plane is transformed so that

$$\int_A\int F(x, y)\, dx\, dy = \int_{A'}\int G(u, v)\, \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv,$$

where A' is the region of the uv-plane corresponding to A. Verify this relation for the transformation from rectangular to polar coordinates.

Find the envelope of each family of curves in Exercises 11–16.

11. The circles $(x - \alpha)^2 + (y - \alpha)^2 = 2\alpha^2$.

12. The lines $x \cos \alpha + y \sin \alpha = 1$.

13. The circles $(x - \cos \alpha)^2 + (y - \sin \alpha)^2 = 1$.

14. The curves $(y - \alpha)^2 = x^2(1 - x)$.

15. The parabolas $x^2 - 2xy + y^2 - 2\alpha x - 2\alpha y + \alpha^2 = 0$.

16. The ellipses $(3 - \cos \alpha)x^2 + 2xy \sin \alpha + (3 + \cos \alpha)y^2 = 4$.

17. Find the envelope of the family of ellipses with coincident axes and of unit area.

18. A line segment of unit length moves in the plane with its ends on the coordinate axes. Find the envelope.

19. A line moves in the plane so that the area formed by it and the coordinate axes is equal to unity. Find the envelope.

20. A line moves in the plane so that the algebraic sum of its intercepts on the coordinate axes is equal to unity. Find the envelope.

Find the envelope of each family of surfaces in Exercises 21–28.

21. The spheres $x^2 + y^2 + (z - \alpha)^2 = 1$.

22. The planes $z = \alpha x + \alpha^2 y$.

23. The planes $\alpha^2 x + y - \alpha z = 0$.

24. The planes $\alpha x + \alpha y + \alpha^2 z = 1$.

25. The planes $(1 - 2\alpha)(x + y - \alpha) + \alpha z = 0$.

26. The planes $\alpha x + \beta y + \alpha\beta z = 4$.

27. The planes $2\alpha x - 2\beta y - 2z = \alpha^2 - \beta^2$.

28. The spheres $(x - \alpha)^2 + (y - \beta)^2 + z^2 = 1$.

29. A plane moves so that the sum of its intercepts on the positive coordinate axes is equal to unity. Find the envelope. (Cf. Exercise 14, Art. 23.)

30. Find the envelope of the family of ellipsoids with coincident axes and of unit volume.

CHAPTER III

ORIGINS OF PARTIAL DIFFERENTIAL EQUATIONS

In this chapter we shall consider some of the ways in which partial differential equations arise. The methods of formulating partial differential equations may be classified under three headings: formal, geometric, and physical processes. In the first method, we begin with a functional relation involving certain arbitrary quantities, which may be either arbitrary constants or arbitrary functions, and eliminate these arbitrary elements between the given relation and those derived from it by partial differentiation. In the second process, we propose a geometric problem, usually by requiring that a certain spatial configuration shall have specified properties, and form one or more relations, among the coordinate variables and partial derivatives, expressing these geometric conditions. In the third type of question, we propose a physical problem whose variables are or depend upon time or position or both, and express a physical law involving these variables and their rates of change.

In each of these cases, there will usually arise the accompanying problem of solving the partial differential equation or equations to which we are led. It will be the concern of later chapters to deal with this sort of problem. Many of the partial differential equations formulated in this chapter will be considered again subsequently.

27. Elimination of arbitrary constants. Consider first a relation containing two independent variables x and y, a dependent variable z, and two arbitrary constants a and b:

$$f(x, y, z, a, b) = 0. \tag{1}$$

By partial differentiation of equation (1), with respect to x and y in turn, we get two additional equations,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p = 0, \tag{2}$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q = 0, \tag{3}$$

where

$$p = \frac{\partial z}{\partial x}, \qquad q = \frac{\partial z}{\partial y}.$$

In general, the arbitrary constants a and b may be eliminated from the set of equations (1), (2), and (3), thereby getting a partial differential equation of the first order:

$$F(x, y, z, p, q) = 0. \tag{4}$$

We call (4) the *eliminant* corresponding to the given relation (1).

Example 1. Obtain a partial differential equation of the first order by eliminating the constants a and b from the equation

$$axz + byz + abxy = 0.$$

Solution. Differentiation with respect to x and y in turn yields two further relations,

$$a(xp + z) + byp + aby = 0,$$

$$axq + b(yq + z) + abx = 0.$$

We may solve two of these three equations for a and b and substitute in the third, or we may find the eliminant by setting equal to zero the determinant of the coefficients of a, b, and ab. The latter procedure gives us

$$\begin{vmatrix} xz & yz & xy \\ xp + z & yp & y \\ xq & yq + z & x \end{vmatrix} = 0,$$

which, upon expansion and reduction, leads to the desired eliminant,

$$xp + yq = z.$$

If a given functional relation involves only one arbitrary constant, two distinct first order partial differential equations may usually be found as eliminants.

Example 2. Find two eliminants corresponding to the equation

$$z = ax + \frac{y}{a}.$$

Solution. Differentiation with respect to x gives us $p = a$, whence elimination leads to the partial equation

$$xp^2 - zp + y = 0.$$

On the other hand, if we differentiate the given equation with respect to y, we get $q = 1/a$, which yields the eliminant

$$yq^2 - zq + x = 0.$$

If the original relation involves more than two arbitrary constants, together with three variables, we must make use of derived equations containing partial derivatives of order higher than the first. Since there are two distinct partial derivatives of the first order, three of second order, $\cdots$, $n + 1$ of nth order, there may be as many as

$$2 + 3 + 4 + \cdots + (n + 1) = \frac{n(n + 3)}{2}$$

arbitrary constants in a relation having as eliminant a partial equation of order n. If the number of arbitrary constants is between $\frac{1}{2}(n - 1)(n + 2)$ and $\frac{1}{2}n(n + 3)$, there will exist, in general, more than one eliminant containing nth partial derivatives.

Example 3. Find the eliminant corresponding to the relation

$$z = a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y.$$

Solution. Partial differentiation gives us

$$p = 2a_1x + a_2y + a_4, \qquad q = a_2x + 2a_3y + a_5,$$

$$r = 2a_1, \qquad s = a_2, \qquad t = 2a_3.$$

Hence

$$a_1 = \frac{r}{2}, \qquad a_2 = s, \qquad a_3 = \frac{t}{2}, \qquad a_4 = p - xr - ys, \qquad a_5 = q - xs - yt,$$

and substitution in the given relation yields the eliminant

$$x^2r + 2xys + y^2t - 2xp - 2yq + 2z = 0,$$

a partial differential equation of the second order.

EXERCISES

Find all possible eliminants of lowest order corresponding to each of the following relations. The letters a, b, c denote arbitrary constants to be eliminated.

1. $z = ax + by$.
2. $z = a(x + y)$.
3. $z = ax^2 + by^2$.
4. $x^2 + y^2 = az$.
5. $(x - a)^2 + (y - b)^2 = z$.
6. $(x - a)^2 + (y - a)^2 = bz$.
7. $(x - a)^2 + (y - b)^2 + z^2 = 1$.
8. $(x - a)^2 + (y - a)^2 + z^2 = 1$.
9. $z = axy + b$.
10. $ax^2 + by^2 + z^2 = 0$.
11. $ax^2 + by^2 + z^2 = 1$.
12. $ax^2 + by^2 = xyz$.
13. $z^2 = (x + a)(y + b)$.
14. $z = ae^{bx} \sin by$.
15. $ax + by + cz = 1$.
16. $z = a_1x^2 + a_2xy + a_3y^2 + a_4$.

17. $(x - a)^2 + (y - b)^2 + (z - c)^2 = 1.$
18. $(x - a_1)^2 + (y - a_2)^2 + (z - a_3)^2 = a_4.$
19. $z = a_1x^2 + a_2y^2 + a_3x + a_4y + a_5.$
20. $z = ax_1 + bx_2 + cx_3.$

28. Elimination of arbitrary functions. We consider next a relation involving an arbitrary function of a specified expression in the variables. We first deal with an example to illustrate the process leading to the eliminant, and later generalize our result.

Example 1. Find the eliminant, as a partial differential equation, corresponding to the relation

$$z = xf\left(\frac{y}{x}\right),$$

where f is an arbitrary function of its argument y/x.

Solution. For convenience, and to emphasize the procedure, denote by u the argument y/x, so that the given relation is expressed by

$$z = xf(u), \qquad u = \frac{y}{x}.$$

Differentiate with respect to x and y in turn, using the symbol $f'(u)$ for $df(u)/du$; we get

$$p = xf'(u)\frac{\partial u}{\partial x} + f(u) = -\frac{y}{x}f'(u) + f(u),$$

$$q = xf'(u)\frac{\partial u}{\partial y} = f'(u).$$

Elimination of $f(u)$ and $f'(u)$ from the given and the two derived equations gives us the desired result,

$$xp + yq = z.$$

We note, in passing, that the above eliminant is the same as that obtained in Example 1 of Art. 27. It is therefore natural to expect that there exist some connection between the two original relations. We have, in fact, from the relation $axz + byz + abxy = 0$ of Example 1, Art. 27,

$$z = -\frac{abxy}{ax + by} = x\left(-\frac{abu}{a + bu}\right), \qquad u = \frac{y}{x},$$

so that this is a special case of the above example with $f(u)$ taken as the indicated function, $-abu/(a + bu)$.

The given relation of our present example is evidently one of the form

$$v = f(u), \tag{1}$$

where u and v are definite expressions in the variables, and f is arbitrary; thus, we had $u = y/x$, $v = z/x$ in our example. In the same manner, we may treat other explicit relations of the form (1), or the equivalent implicit relation

$$\phi(u, v) = 0, \tag{2}$$

where u and v are specified, as before, and ϕ is arbitrary. Because of its importance in later work, we take as our next illustration of the process of elimination of arbitrary functions, the following general example.

Example 2. Find the eliminant corresponding to relation (2), u and v being given functionally independent expressions in x, y, z, and ϕ being arbitrary. Regard z as the dependent variable.

Solution. Differentiating (2) partially with respect to x and y in turn, we get (Arts. 18, 20)

$$\begin{aligned}\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)&=0,\\ \frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)&=0.\end{aligned} \tag{3}$$

These are two linear homogeneous algebraic equations in $\partial\phi/\partial u$ and $\partial\phi/\partial v$. Since these quantities are not both to be zero, the determinant of the system must vanish. Hence we have

$$\begin{vmatrix} u_x + u_z p & v_x + v_z p \\ u_y + u_z q & v_y + v_z q \end{vmatrix} = 0,$$

which, upon expansion and reduction, gives us

$$(u_y v_z - u_z v_y)p + (u_z v_x - u_x v_z)q = u_x v_y - u_y v_x, \tag{4}$$

a linear relation in p and q. This partial differential equation of first order is the desired eliminant corresponding to (2). It is readily verified that the relation of Example 1 and its eliminant constitute a special case of the above result.

The coefficients in (4) are evidently the Jacobians (Art. 24)

$$\frac{\partial(u, v)}{\partial(y, z)} = u_y v_z - u_z v_y, \quad \frac{\partial(u, v)}{\partial(z, x)} = u_z v_x - u_x v_z, \quad \frac{\partial(u, v)}{\partial(x, y)} = u_x v_y - u_y v_x. \tag{5}$$

We have stipulated that u and v be functionally independent, for if they were dependent, each of the Jacobians (5) would vanish identically, and (4) would reduce to the triviality $0 = 0$.

When a given relation involves more than one arbitrary function, the elimination process is not usually so straightforward. In general, an equation containing n arbitrary functions leads to more than one eliminant, in which partial derivatives of order greater than n appear. In some cases, however, a single eliminant, involving derivatives of order n at most, is obtainable.

Example 3. Find the eliminant of lowest order corresponding to the relation

$$z = f(x + ay) + g(x - ay),$$

where f and g are arbitrary functions and a is a fixed constant.

Solution. Let $u = x + ay$, $v = x - ay$, so that the given relation may be written compactly as

$$z = f(u) + g(v).$$

Then we have

$$p = f'(u) + g'(v), \qquad q = af'(u) - ag'(v).$$

It is apparent that the four quantities f, g, f', g' cannot be eliminated from the three equations at our disposal. Accordingly we find second derivatives; these are

$$r = f''(u) + g''(v), \qquad s = af''(u) - ag''(v), \qquad t = a^2f''(u) + a^2g''(v).$$

It now happens that only f'' and g'' appear in these three equations, and consequently the eliminant can be found. Evidently we have

$$t = a^2r$$

as the desired eliminant.

EXERCISES

Find all possible eliminants of lowest order corresponding to each of the following relations. The letters f, g, h denote arbitrary functions to be eliminated.

1. $z = xf(y)$.
2. $z = yf(x)$.
3. $z = (x + y)f(xy)$.
4. $x + y = f(xyz)$.
5. $(x - y)z = f(xz + yz)$.
6. $z = f(x^2 + y^2)$.
7. $xyz = f(x + y + z)$.
8. $z = xyf\left(\dfrac{x + y}{z}\right)$.
9. $xyz = f(x^2 + y^2 + z^2)$.
10. $xy + yz + zx = f\left(\dfrac{z}{x + y}\right)$.
11. $z = f(x) + g(y) + xy$.
12. $z = f(3x + 2y) + g(2x + 3y)$.
13. $z = yf(x) + g(x)$.
14. $z = x^2f(y) + xg(y)$.
15. $z = f(x)\cdot g(y)$.
16. $z = f(x) + e^{xy}g(x)$.
17. $z = f(y) + g(x + y + z)$.
18. $z = xf(y) + yg(x)$.
19. $z = f(y) + xg(y) + x^2h(y)$.
20. $z = f(x) + g(x + y) + h(x - y)$.

29. Geometric problems. Partial differential equations corresponding to specified types of surfaces, satisfying given geometric conditions, may be obtained by either of two general methods. On the one hand, we may express the equation of such surfaces as a relation involving certain arbitrary constants or arbitrary functions which may be eliminated by the procedures outlined in Arts. 27–28; by the other method, we may formulate the partial differential equation directly, using the geometric properties of partial derivatives (Chapter II). We illustrate these methods in the following examples.

Example 1. Find the partial differential equation of all spheres of unit radius and with their centers in the xy-plane.

Solution. (a) The equation of these spheres is readily formed; we have

$$(x - a)^2 + (y - b)^2 + z^2 = 1,$$

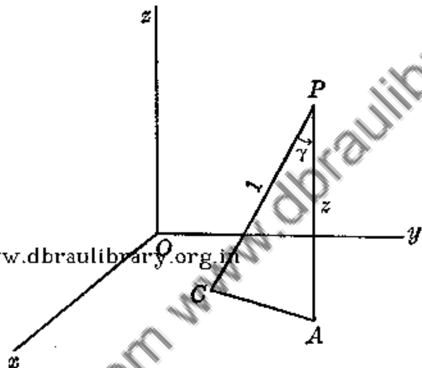

FIG. 7

where $(a, b, 0)$ are the coordinates of the center, a and b being arbitrary constants. By the method of Art. 27 (cf. Exercise 7, Art. 27), we then find as the eliminant,

$$z^2(p^2 + q^2 + 1) = 1.$$

This partial equation of first order is the desired result.

(b) Alternatively, we may proceed as follows. Let P: (x, y, z) be any point on one of the spheres with center C in the xy-plane, and let A be the projection of P on the xy-plane as shown in Fig. 7. Then $CP = 1$ and

$$\cos\gamma = \cos APC = z. \tag{1}$$

Also, since (Art. 21)

$$\frac{\cos\alpha}{p} = \frac{\cos\beta}{q} = \frac{\cos\gamma}{-1},$$

we have

$$\cos\alpha = -p\cos\gamma = -zp, \qquad \cos\beta = -q\cos\gamma = -zq. \tag{2}$$

Hence, using the relation $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, we get from (1) and (2),

$$z^2(p^2 + q^2 + 1) = 1.$$

Example 2. Find the partial differential equation of the surfaces which are such that the normal at any point (x, y, z) is perpendicular to the line through that point and with direction numbers given by the functions $P(x, y, z)$, $Q(x, y, z)$, $R(x, y, z)$.

Solution. The normal to the surface at (x, y, z) has direction numbers $p, q, -1$. Hence we need merely express the fact that the line whose direction is given by P, Q, R is perpendicular to the line with direction numbers $p, q, -1$. The condition for perpendicularity (Art. 22) then gives us immediately

$$Pp + Qq = R. \tag{3}$$

Since this equation is of the same form as equation (4) of Example 2, Art. 28, we may expect a close relation between the problems of that example and the present one. The nature of this relation will be discussed in detail in Chapter IV.

Example 3. Find the partial differential equation of all developable surfaces.

Solution. A developable surface is defined (Art. 26) as the envelope of a one-parameter family of planes. Such a family of planes must then constitute the tangent planes to the surface, and the equation of the planes may be written in the form (Art. 21)

$$Z - z = p(X - x) + q(Y - y), \tag{4}$$

where p, q, x, y, z are all functions of a single parameter. In particular, since p and q are functions of this parameter, there will exist a functional relation between them, say

$$q = f(p). \tag{5}$$

Differentiating (5) with respect to x and y in turn, we get

$$s = f'(p)r, \qquad t = f'(p)s,$$

and elimination of $f'(p)$ between these equations yields the desired relation,

$$rt - s^2 = 0. \tag{6}$$

Equation (6) expresses the fact that the Jacobian of p and q with respect to x and y vanishes identically,

$$\frac{\partial(p, q)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial p}{\partial x} & \dfrac{\partial q}{\partial x} \\ \dfrac{\partial p}{\partial y} & \dfrac{\partial q}{\partial y} \end{vmatrix} = \begin{vmatrix} r & s \\ s & t \end{vmatrix} = rt - s^2 = 0.$$

Conversely, if equation (6) holds for any surface $z = F(x, y)$, there must exist a functional relation between p and q, say $q = f(p)$. We may exclude the case in which two such relations exist, for then p and q would be constants and the surface $z = F(x, y)$ would be linear in x and y, that is, the surface would be a plane. Now

$$\frac{\partial(z - xp - yq, p)}{\partial(x, y)} = \begin{vmatrix} -xr - ys & r \\ -xs - yt & s \end{vmatrix}$$

$$= -xrs - ys^2 + xrs + yrt$$

$$= y(rt - s^2) = 0$$

by (6), and therefore $z - xp - yq$ and p must be functionally dependent. Thus we have, using $q = f(p)$,

$$z - xp - yf(p) = g(p). \tag{7}$$

Differentiating the latter equation with respect to x and y in turn, we find

$$[x + yf'(p) + g'(p)]r = 0, \qquad [x + yf'(p) + g'(p)]s = 0.$$

Since p is not to be constant, $r = \partial p/\partial x$ and $s = \partial p/\partial y$ cannot both vanish, and consequently the relation

$$x + yf'(p) + g'(p) = 0 \tag{8}$$

must hold. Hence the surfaces satisfying (6) must be those obtainable by eliminating p from (7) and (8). But this elimination process is precisely that used for finding the envelope of the one-parameter family of planes (7), p being regarded as parameter. Therefore the partial differential equation (6) is satisfied by the envelope of a one-parameter family of planes, that is, by a developable surface.

EXERCISES

Find the partial differential equation (or equations) of the surfaces described in each of the following exercises.

1. Planes through the origin.
2. Planes parallel to the x-axis.
3. Planes with z-intercept equal to unity.
4. Cylinders with elements parallel to the y-axis.
5. Spheres with centers on the line $x = y = z$.
6. Spheres with unit radius and centers in the plane $x = y$.
7. Spheres with centers in the plane $x = y$ and tangent to the xy-plane.
8. Spheres with centers in the xy-plane and passing through the origin.
9. Surfaces of revolution with the z-axis as axis.
10. Surfaces whose normals meet the xy-plane at an angle of 45°.
11. Surfaces whose normals make an angle of 60° with the x-axis.
12. Surfaces for which the tangent planes have intercepts, on the coordinate axes, whose sum is unity.
13. Surfaces for which the tangent planes have intercepts, on the coordinate axes, whose product is unity.
14. Surfaces for which the tangent planes have intercepts, on the coordinate axes, the sum of whose squares is unity.
15. All planes not parallel to the z-axis.

16. All spheres.

17. Paraboloids with axes parallel to the z-axis and with sections parallel to the xy-plane which are conics with axes parallel to the x- and y-axes.

18. Paraboloids through the origin and with axes parallel to the z-axis.

19. Quadric surfaces with axes parallel to the coordinate axes and with sections parallel to the xy-plane which are circles.

20. Quadric surfaces through the origin and with axes parallel to the coordinate axes.

30. The vibrating string. The remainder of this chapter will be devoted to a few physical problems leading to partial differential equations. Later in the book we shall again discuss the partial equations formulated here, and shall solve them subject to suitable physical conditions.

Consider first a string stretched taut between two fixed points a distance L (ft.) apart. For definiteness, we take these two points at the origin O and at $(L, 0)$ on the x-axis, as shown in Fig. 8. Let F (lb.) be

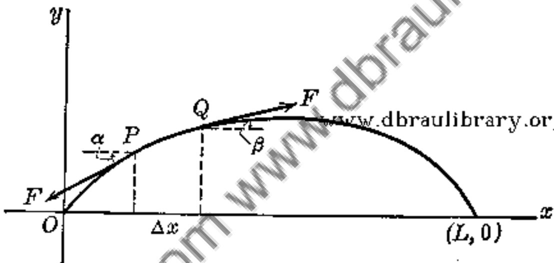

FIG. 8

the constant tension at any point of the string, and let w (lb./ft.) be the weight of the string per unit length. If the string is set vibrating, in some manner we need not now specify, in the xy-plane, the subsequent displacement y (ft.) from the equilibrium position of a point P of the string will be a function of distance x (ft.) and of time t (sec.).

We assume (a) that the constant tension F is so large compared with the weight wL that the gravitational force may be neglected; (b) that the displacement y of any point is so small compared with L that the length of the string may be taken as L for each of its positions; and (c) that each point P of the string traverses a straight line parallel to the y-axis, that is, that the vibrations are purely transverse.

The force (lb.) acting on a segment PQ, originally of length Δx (ft.), is, by Newton's second law of motion,

$$\frac{w}{g}\,\Delta x\,\frac{\partial^2 y(x_1, t)}{\partial t^2}\,, \tag{1}$$

where the acceleration $\partial^2 y/\partial t^2$ (ft./sec.^{2}) is taken at some point between P and Q, $x < x_1 < x + \Delta x$, and $g = 32.2$ ft./sec.^{2} This force is the resultant of the y-components of tension at the extremities P and Q; if α and β are the angles between the tension vectors and the x-axis at the two ends, as shown in the figure, we have for the force producing the acceleration,

$$F(\sin \beta - \sin \alpha). \tag{2}$$

Now

$$\sin \alpha = \frac{\tan \alpha}{\sqrt{1 + \tan^2 \alpha}}, \qquad \sin \beta = \frac{\tan \beta}{\sqrt{1 + \tan^2 \beta}}$$

where

$$\tan \alpha = \frac{\partial y(x, t)}{\partial x}, \qquad \tan \beta = \frac{\partial y(x + \Delta x, t)}{\partial x}$$

are the slopes at P and Q, respectively. By assumption (b), restricting the string to small vibrations, these slopes will be small, and their squares may be neglected in comparison with unity. Consequently we have as a good approximation,

$$F(\sin \beta - \sin \alpha) = F\left[\frac{\partial y(x + \Delta x, t)}{\partial x} - \frac{\partial y(x, t)}{\partial x}\right]. \tag{3}$$

Equating the latter expression to (1), and dividing by $w\,\Delta x/g$, we get

$$\frac{\partial^2 y(x_1, t)}{\partial t^2} = a^2 \frac{\dfrac{\partial y(x + \Delta x, t)}{\partial x} - \dfrac{\partial y(x, t)}{\partial x}}{\Delta x}, \tag{4}$$

where $a = \sqrt{Fg/w}$ (ft./sec.) and $x < x_1 < x + \Delta x$. If we now allow Δx to approach zero, whence x_1 approaches x, the difference quotient on the right of (4) approaches the partial derivative of $\partial y/\partial x$ with respect to x. Hence, in the limit, we get

$$\frac{\partial^2 y(x, t)}{\partial t^2} = a^2 \frac{\partial^2 y(x, t)}{\partial x^2}. \tag{5}$$

This partial differential equation of the second order is that of the vibrating string.

EXERCISES

1. Verify the fact that

$$y(x, t) = \sin \frac{n\pi x}{L} \cos \frac{n\pi a t}{L},$$

where n is an integer, is a solution of the equation (5) of the vibrating string. Hence show that the frequency of this vibration is proportional to the square root of the tension F and inversely proportional to the length L and the square root of the linear density w.

2. Show that

$$y(x, t) = f(x + at) + \phi(x - at),$$

where f and ϕ are arbitrary functions of their respective arguments, is a solution of equation (5). (Cf. Example 3, Art. 28.) Interpret this solution as the resultant of two wave motions traveling in opposite directions with speed a.

3. Assuming that the linear density and the tension are functions of distance x along the string, say $w(x)$ and $F(x)$, show that the partial differential equation of the vibrating string becomes

$$\frac{w(x)}{g}\frac{\partial^2 y}{\partial t^2} = F(x)\frac{\partial^2 y}{\partial x^2} + \frac{dF}{dx}\frac{\partial y}{\partial x}.$$

4. Consider a vibrating membrane at each point of which there is a constant tension F (lb./ft.) acting in all directions. Let δ (lb./ft.^{2}) be the surface density of the material, and take the equilibrium position of the membrane to be the xy-plane. Show that, under suitable assumptions, the displacement z (ft.) at any point (x, y) and at any time t (sec.) is given by

$$\frac{\partial^2 z}{\partial t^2} = a^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

where $a = \sqrt{Fg/\delta}$ (ft./sec.).

5. Consider the longitudinal vibrations of a long homogeneous elastic rod of density ρ (lb./ft.^{3}), constant cross-sectional area A (ft.^{2}), and modulus of elasticity E (lb./ft.^{2}). Show that the displacement X (ft.) of a point initially x ft. from one end and at time t (sec.) is given by

$$\frac{\partial^2 X}{\partial t^2} = a^2\frac{\partial^2 X}{\partial x^2},$$

where $a = \sqrt{Eg/\rho}$ (ft./sec.).

31. One-dimensional heat flow. In this and the next article we shall consider the flow of heat and the accompanying variation of temperature with position and with time. The following empirical laws form the basis of the analysis.

(a) The quantity of heat in a small portion of a body is proportional to its mass and to its temperature.

(b) Heat flows from a higher to a lower temperature.

(c) The rate of flow across an area is proportional to the area and to the temperature gradient (that is, the rate of change of temperature with respect to distance measured normal to the area) at a point of the area.

Consider a bar or rod of homogeneous material of density δ (gr./cm.^{3}) and having a constant cross-sectional area A(cm.^{2}). We suppose that the sides of the bar are insulated so that the streamlines of heat flow are parallel, and perpendicular to the area A. Take the distance x (cm.) from one end of the bar and positive in the direction of flow.

By law (a), the quantity of heat in a lamina of thickness Δx is $c\delta A\ \Delta x(\tau + 273)$ calories, where τ is the temperature (° C.) of the lamina and the constant of proportionality c is called the specific heat (cal./gr. deg.). Then if R_1 and R_2 are respectively the rates (cal./sec.) of inflow and outflow, for the sections $x = x$ and $x = x + \Delta x$,

$$c\delta A\ \Delta x \frac{\partial \tau(x_1, t)}{\partial t} = R_1 - R_2, \tag{1}$$

where $x < x_1 < x + \Delta x$, and t (sec.) is time. By law (c), we also have

$$R_1 = -KA \frac{\partial \tau(x, t)}{\partial x}, \qquad R_2 = -KA \frac{\partial \tau(x + \Delta x, t)}{\partial x}, \tag{2}$$

where the negative signs appear as a consequence of law (b), and where K is called the thermal conductivity (cal./cm. deg. sec.). Combining (1) and (2), and dividing by $c\delta A\ \Delta x$, we get

$$\frac{\partial \tau(x_1, t)}{\partial t} = \frac{K}{c\delta} \frac{\dfrac{\partial \tau(x + \Delta x, t)}{\partial x} - \dfrac{\partial \tau(x, t)}{\partial x}}{\Delta x}. \tag{3}$$

Now let Δx approach zero, so that x_1 approaches x. Then the difference quotient in the right member of (3) approaches the partial derivative of $\partial \tau/\partial x$ with respect to x, and there is found

$$\frac{\partial \tau(x, t)}{\partial t} = \alpha^2 \frac{\partial^2 \tau}{\partial x^2}, \tag{4}$$

where $\alpha^2 = K/c\delta$ is the diffusivity (cm.²/sec.). Equation (4) is the partial differential equation for the temperature in the case of one-dimensional heat flow.

EXERCISES

1. For one-dimensional heat flow under steady-state conditions, in which the temperature at any point is independent of time, show that the temperature is a linear function of distance. Hence show that the temperature under steady-state conditions at any point of a bar of length L (cm.) is

$$\tau = \frac{\tau_L - \tau_0}{L} x + \tau_0,$$

where τ_0 and τ_L are the respective temperatures at the ends.

2. Verify the fact that

$$\tau = \sin \frac{n\pi x}{L} e^{-n^2\pi^2\alpha^2 t/L^2},$$

where n is an integer, is a solution of equation (4). Describe the initial temperature distribution (for $t = 0$) in a bar of length L (cm.), and the subsequent boundary temperatures (for $x = 0$ and $x = L$).

3. Suppose that the density δ, the area A, the specific heat c, and the thermal conductivity K are functions of distance x but are constant throughout any particular cross-section. Show that the equation of one-dimensional heat flow then is

$$c(x)\delta(x)A(x)\frac{\partial \tau}{\partial t} = K(x)A(x)\frac{\partial^2 \tau}{\partial x^2} + \left[K(x)\frac{dA}{dx} + A(x)\frac{dK}{dx}\right]\frac{\partial \tau}{\partial x}.$$

4. Consider a long cylindrical pipe of length L (cm.). Suppose the temperature at every point of the inner surface to be the same, and that at every point of the outer surface to be the same, at a particular time t, but that each surface temperature changes with time. Then heat will flow radially, and the temperature on any cylindrical surface of area $A = 2\pi rL$, where r (cm.) is the distance from the center, is constant. Show that the rate of heat flow is

$$\frac{\partial Q}{\partial t} = -2\pi KLr\frac{\partial \tau}{\partial r}$$

and that the temperature τ is given by

$$\frac{\partial \tau}{\partial t} = \alpha^2\left(\frac{\partial^2 \tau}{\partial r^2} + \frac{1}{r}\frac{\partial \tau}{\partial r}\right).$$

5. Consider a spherical shell with surface temperature conditions similar to those of the pipe of Exercise 4. Show that the rate of heat flow is

$$\frac{\partial Q}{\partial t} = -4\pi Kr^2\frac{\partial \tau}{\partial r},$$

where r (cm.) is the distance from the center, and that the temperature τ is given by

$$\frac{\partial \tau}{\partial t} = \alpha^2\left(\frac{\partial^2 \tau}{\partial r^2} + \frac{2}{r}\frac{\partial \tau}{\partial r}\right).$$

32. Heat flow in space. We now extend the argument of Art. 31 to the case of three-dimensional flow, which will include two-dimensional flow as a special case. To do this, we use orthogonal curvilinear coordinates (Art. 23).

Let $\tau(u, v, w, t)$ be the temperature at any body position with orthogonal curvilinear coordinates (u, v, w) and at any time t. Let δ, c, and K, respectively, denote the density, specific heat, and thermal conductivity, assumed constant, as before. For the typical element of volume, with edges $E_1\,\Delta u$, $E_2\,\Delta v$, $E_3\,\Delta w$, as shown in Fig. 9, we may consider the rate of flow into and out of each pair of opposite faces.

For the two faces normal to the u-direction, with the edges $E_2\,\Delta v$ and $E_3\,\Delta w$, the temperature gradients will be the respective values of the limit of $\Delta\tau/(E_1\,\Delta u)$ at two points (u, v_1, w_1) and $(u + \Delta u, v_1, w_1)$, where $v < v_1 < v + \Delta v$, $w < w_1 < w + \Delta w$. Hence the rate of gain and rate of loss through these faces will be approximately

$$-K\left[E_2E_3\,\Delta v\,\Delta w\,\frac{1}{E_1}\frac{\partial\tau}{\partial u}\right]_{(u,\,v_1,\,w_1)}$$

and

$$-K\left[E_2E_3\,\Delta v\,\Delta w\,\frac{1}{E_1}\frac{\partial\tau}{\partial u}\right]_{(u+\Delta u,\,v_1,\,w_1)}$$

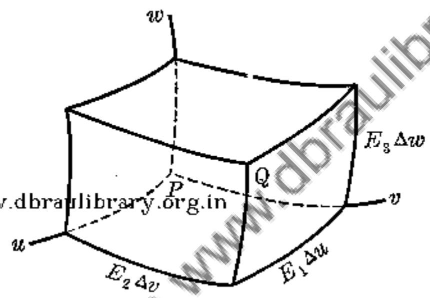

FIG. 9

respectively, so that the net rate of gain for the u-direction is

$$K\,\Delta u\,\Delta v\,\Delta w\,\frac{\left[\dfrac{E_2E_3}{E_1}\dfrac{\partial\tau}{\partial u}\right]_{(u+\Delta u,\,v_1,\,w_1)} - \left[\dfrac{E_2E_3}{E_1}\dfrac{\partial\tau}{\partial u}\right]_{(u,\,v_1,\,w_1)}}{\Delta u}. \qquad (1)$$

Similarly, the net rates of gain for the other two pairs of faces, in the v- and w-directions, will be approximately given by

$$K\,\Delta u\,\Delta v\,\Delta w\,\frac{\left[\dfrac{E_1E_3}{E_2}\dfrac{\partial\tau}{\partial v}\right]_{(u_2,\,v+\Delta v,\,w_2)} - \left[\dfrac{E_1E_3}{E_2}\dfrac{\partial\tau}{\partial v}\right]_{(u_2,\,v,\,w_2)}}{\Delta v} \qquad (2)$$

and

$$K\,\Delta u\,\Delta v\,\Delta w\,\frac{\left[\dfrac{E_1E_2}{E_3}\dfrac{\partial\tau}{\partial w}\right]_{(u_3,\,v_3,\,w+\Delta w)} - \left[\dfrac{E_1E_2}{E_3}\dfrac{\partial\tau}{\partial w}\right]_{(u_3,\,v_3,\,w)}}{\Delta w}, \qquad (3)$$

where each expression in brackets is evaluated at a suitable interior point.

On the other hand, the rate of gain of heat for the element, whose volume is approximately equal to $E_1E_2E_3 \; \Delta u \; \Delta v \; \Delta w$, is also given nearly by

$$c\delta \; \Delta u \; \Delta v \; \Delta w \left[E_1E_2E_3 \frac{\partial \tau}{\partial t} \right]_{(u_4,\; v_4,\; w_4)}, \tag{4}$$

where $u < u_4 < u + \Delta u, \quad v < v_4 < v + \Delta v, \quad w < w_4 < w + \Delta w.$

Therefore, equating the expression (4) to the sum of (1), (2), and (3), dividing by $\Delta u \; \Delta v \; \Delta w$, and passing to the limit as Δu, Δv, Δw all approach zero, we get

$$E_1E_2E_3 \frac{\partial \tau}{\partial t} = \alpha^2 \left[\frac{\partial}{\partial u}\left(\frac{E_2E_3}{E_1} \frac{\partial \tau}{\partial u} \right) + \frac{\partial}{\partial v}\left(\frac{E_1E_3}{E_2} \frac{\partial \tau}{\partial v} \right) + \frac{\partial}{\partial w}\left(\frac{E_1E_2}{E_3} \frac{\partial \tau}{\partial w} \right) \right], \tag{5}$$

where $\alpha^2 = K/c\delta$ is the diffusivity.

In the simplest case, the coordinate system is rectangular, for then $E_1 = E_2 = E_3 = 1$ (Art. 23). Equation (5) then becomes, with $u = x$, $v = y$, $w = z$,

$$\frac{\partial \tau}{\partial t} = \alpha^2 \left(\frac{\partial^2 \tau}{\partial x^2} + \frac{\partial^2 \tau}{\partial y^2} + \frac{\partial^2 \tau}{\partial z^2} \right). \tag{6}$$

In particular, if the streamlines of heat flow are curves in parallel planes, so that the temperature does not vary with, say, z, (6) reduces to

$$\frac{\partial \tau}{\partial t} = \alpha^2 \left(\frac{\partial^2 \tau}{\partial x^2} + \frac{\partial^2 \tau}{\partial y^2} \right). \tag{7}$$

This is the rectangular form of the partial differential equation of two-dimensional flow. If, further, the streamlines are straight lines, say parallel to the x-axis, (7) then reduces to the equation obtained in Art. 31.

When the temperature at each point of the body is independent of time, the flow is said to be in the steady state. Equation (6) then takes the form

$$\frac{\partial^2 \tau}{\partial x^2} + \frac{\partial^2 \tau}{\partial y^2} + \frac{\partial^2 \tau}{\partial z^2} = 0. \tag{8}$$

This is the three-dimensional rectangular form of an important partial differential equation called *Laplace's equation.* In two dimensions, Laplace's equation is

$$\frac{\partial^2 \tau}{\partial x^2} + \frac{\partial^2 \tau}{\partial y^2} = 0. \tag{9}$$

We shall meet this equation again in our later work.

EXERCISES

1. Using the transformation $x = \rho \cos \theta$, $y = \rho \sin \theta$ from rectangular to polar coordinates, show that equation (7) is changed into

$$\frac{\partial \tau}{\partial t} = \alpha^2 \left(\frac{\partial^2 \tau}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \tau}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \tau}{\partial \theta^2} \right).$$

Hence deduce Laplace's equation in two-dimensional polar form.

2. Transform from rectangular to cylindrical space coordinates, using the relations given in Art. 23 to determine E_1, E_2, and E_3, and show that equation (5) becomes

$$\frac{\partial \tau}{\partial t} = \alpha^2 \left(\frac{\partial^2 \tau}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \tau}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \tau}{\partial \theta^2} + \frac{\partial^2 \tau}{\partial z^2} \right).$$

3. Transform from rectangular to spherical coordinates (Art. 23), and show that equation (5) becomes

$$\frac{\partial \tau}{\partial t} = \alpha^2 \left(\frac{\partial^2 \tau}{\partial \rho^2} + \frac{2}{\rho} \frac{\partial \tau}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \tau}{\partial \phi^2} + \frac{\cot \phi}{\rho^2} \frac{\partial \tau}{\partial \phi} + \frac{\csc^2 \phi}{\rho^2} \frac{\partial^2 \tau}{\partial \theta^2} \right).$$

4. Using the result of Exercise 2, obtain the partial differential equation of Exercise 4, Art. 31.

5. Using the result of Exercise 3, obtain the partial differential equation of Exercise 5, Art. 31.

33. Flow of electricity in a cable. We consider next the flow of electricity in a cable having constant resistance, inductance, capacitance, and leakance. Let $x = OP$ (miles) be the distance along the line from the source as origin O, and let

$E(x, t)$ = potential (volts) at P at time t (sec.),
$I(x, t)$ = current (amp.) at P at time t,
R = resistance (ohms/mile),
L = inductance (henries/mile),
C = capacitance (farads/mile),
G = leakance (mhos/mile).

The drop in potential along a segment $\Delta x = PQ$ (mi.) will be approximately

$$\Delta E = -R\Delta x \cdot I - L\Delta x \frac{\partial I(x_1, t)}{\partial t},$$

where $x < x_1 < x + \Delta x$. Dividing by Δx and allowing it to approach zero, we get in the limit

$$\frac{\partial E(x, t)}{\partial x} = -RI - L\frac{\partial I(x, t)}{\partial t}. \tag{1}$$

Likewise, the drop in current along Δx is approximately

$$\Delta I = -G\Delta x \cdot E - C\Delta x \frac{\partial E(x_2, t)}{\partial t},$$

whence

$$\frac{\partial I(x, t)}{\partial x} = -GE - C\frac{\partial E(x, t)}{\partial t}. \tag{2}$$

Equations (1) and (2) give us a system of simultaneous partial differential equations with the independent variables x and t and the dependent variables E and I. To eliminate I, differentiate (1) partially with respect to x, getting

$$\frac{\partial^2 E}{\partial x^2} = -R\frac{\partial I}{\partial x} - L\frac{\partial^2 I}{\partial x \partial t}, \tag{3}$$

and differentiate (2) partially with respect to t, so that

$$\frac{\partial^2 I}{\partial t \partial x} = -G\frac{\partial E}{\partial t} - C\frac{\partial^2 E}{\partial t^2}. \tag{4}$$

Elimination of $\partial I/\partial x$ and $\partial^2 I/\partial t\partial x = \partial^2 I/\partial x \partial t$ from (2), (3), and (4) gives us

$$\frac{\partial^2 E}{\partial x^2} = RGE + (RC + LG)\frac{\partial E}{\partial t} + LC\frac{\partial^2 E}{\partial t^2}. \tag{5}$$

Similarly, elimination of E yields

$$\frac{\partial^2 I}{\partial x^2} = RGI + (RC + LG)\frac{\partial I}{\partial t} + LC\frac{\partial^2 I}{\partial t^2}. \tag{6}$$

Thus the potential E and current I satisfy the same equation.

Equations (1), (2), (5), and (6) are sometimes called the *telephone equations.*

EXERCISES

1. Under steady-state conditions, in which the potential and current are independent of the time, show that the telephone equations are satisfied by a solution of the form

$$E = c_1 e^{\sqrt{RG}\,x} + c_2 e^{-\sqrt{RG}\,x}, \qquad I = -\sqrt{\frac{G}{R}}\left(c_1 e^{\sqrt{RG}\,x} - c_2 e^{-\sqrt{RG}\,x}\right),$$

where c_1 and c_2 are arbitrary constants. Show how to determine proper values of c_1 and c_2 when the potential and current at the source are given.

2. If the leakance and inductance of a transmission line are so small that their effects can be neglected in comparison with the effects of resistance and capacitance, show that the telephone equations reduce to

$$\frac{\partial E}{\partial x} = -RI, \qquad \frac{\partial I}{\partial x} = -C\frac{\partial E}{\partial t}, \qquad \frac{\partial^2 E}{\partial x^2} = RC\frac{\partial E}{\partial t}, \qquad \frac{\partial^2 I}{\partial x^2} = RC\frac{\partial I}{\partial t}.$$

These are called the *telegraph equations*. Note that the last two are of the same form as the equation of one-dimensional heat flow (Art. 31).

3. If the effects of resistance and leakance are small compared with the effects of inductance and capacitance, so that the former constants are negligible, show that the telephone equations reduce to

$$\frac{\partial E}{\partial x} = -L\frac{\partial I}{\partial t}, \qquad \frac{\partial I}{\partial x} = -C\frac{\partial E}{\partial t}, \qquad \frac{\partial^2 E}{\partial x^2} = LC\frac{\partial^2 E}{\partial t^2}, \qquad \frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2}.$$

These are called the *radio equations*, useful under high frequency conditions. Note that they are of the same form as the equation of the vibrating string (Art. 30).

4. Show that the radio equations of Exercise 3 have the solutions

$$E = f\left(x + \frac{t}{\sqrt{LC}}\right) + \phi\left(x - \frac{t}{\sqrt{LC}}\right),$$

$$I = -\sqrt{\frac{C}{L}}\left[f\left(x + \frac{t}{\sqrt{LC}}\right) - \phi\left(x - \frac{t}{\sqrt{LC}}\right)\right],$$

where f and ϕ are arbitrary. (Cf. Exercise 2, Art. 30.)

5. If $RC = LG$, the transmission line is said to be *distortionless*. Show that under this condition the transformation of variables

$$E = E_1 e^{-Rt/L}, \qquad I = I_1 e^{-Rt/L}$$

changes the telephone equations into the form of the radio equations of Exercise 3.

34. Fluid flow. As our final example of a physical problem leading to partial differential equations, we consider so-called perfect fluid flow, in which the fluid is supposed to be homogeneous and continuous in structure and for which viscosity may be neglected. These assumptions imply that small elements of a fluid body have the same physical properties as the entire body, and that all forces exerted by the fluid on an immersed surface may be taken normal to the surface whether that surface is at rest or in motion in the fluid.

We investigate three-dimensional flow relative to a set of rectangular coordinate axes, where the coordinates (x, y, z) are measured in feet. Let u, v, w (ft./sec.) be the components of velocity in the x-, y-, and z-directions respectively, at a point $P:(x, y, z)$ at time t (sec.), so that u, v, w are functions of the four variables x, y, z, t. In a time interval Δt, the fluid particle originally at P will have moved to another point $Q:(x + \Delta x, y + \Delta y, z + \Delta z)$. Consider this one moving particle, so that its coordinates (x, y, z) will be functions of t. Then the components of acceleration of this particle are

$$\begin{aligned} \frac{du}{dt} &= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} + \frac{\partial u}{\partial t} \\ &= u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}, \\ \frac{dv}{dt} &= u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}, \\ \frac{dw}{dt} &= u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}. \end{aligned} \tag{1}$$

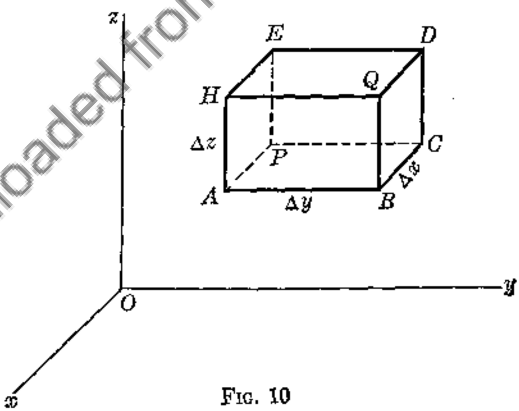

FIG. 10

Now let $P:(x, y, z)$ and $Q:(x + \Delta x, y + \Delta y, z + \Delta z)$ be the opposite corners of a small rectangular parallelepiped with edges Δx, Δy, and Δz, as shown in Fig. 10, and let $\delta(x, y, z, t)$ (lb./ft.^{3}) be the density at P at any time t. Suppose the fluid to be in motion under the action of a force (lb.) per unit weight (lb.), say $G(x, y, z, t)$, whose components in the x-, y-, and z-directions are respectively G_1, G_2, and G_3, and let $F(x, y, z, t)$ (lb./ft.^{2}) be the fluid pressure at P at time t. Then, by Newton's second law of motion, we have for the x-direction the approximate relation

$$\left[\frac{\delta}{g}\,\Delta x\,\Delta y\,\Delta z\,\frac{du}{dt}\right]_{(x_1,\,y_1,\,z_1)} = [G_1\delta\,\Delta x\,\Delta y\,\Delta z]_{(x_2,\,y_2,\,z_2)} + [F\,\Delta y\,\Delta z]_{(x,\,y_3,\,z_3)} - [F\,\Delta y\,\Delta z]_{(x+\Delta x,\,y_3,\,z_3)},$$

where $g = 32.2$ ft./sec.^{2} and each bracket is evaluated at a suitable interior point. If we divide this equation throughout by

$$[\delta\,\Delta x\,\Delta y\,\Delta z/g]_{(x_1,\,y_1,\,z_1)},$$

and allow Δx, Δy, and Δz to approach zero, we get in the limit, at the point P,

$$\frac{du}{dt} = gG_1 - \frac{g}{\delta}\frac{\partial F}{\partial x}. \tag{2}$$

Combining (2) and the first of equations (1), we have

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} = gG_1 - \frac{g}{\delta}\frac{\partial F}{\partial x}. \tag{3}$$

Similarly, we get for the y- and z-directions,

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} = gG_2 - \frac{g}{\delta}\frac{\partial F}{\partial y}, \tag{4}$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} = gG_3 - \frac{g}{\delta}\frac{\partial F}{\partial z}. \tag{5}$$

Equations (3), (4), (5) are the partial differential equations of three-dimensional fluid flow.

For two-dimensional flow, say in planes parallel to the xy-plane, equations (3)–(5) reduce to

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t} = gG_1 - \frac{g}{\delta}\frac{\partial F}{\partial x}, \tag{6}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial v}{\partial t} = gG_2 - \frac{g}{\delta}\frac{\partial F}{\partial y}; \tag{7}$$

and for one-dimensional flow, say in the x-direction, these equations further reduce to

$$u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = gG - \frac{g}{\delta}\frac{\partial F}{\partial x}. \tag{8}$$

In addition to the above equations of motion, there exists another relation among the variables, known as the equation of continuity. This may be obtained as follows. The rate at which fluid enters the face $PCDE$ (Fig. 10) is approximately $[\delta u\ \Delta y\ \Delta z]_{(x,\ y_4,\ z_4)}$ (lb./sec.), and the rate at which it leaves the opposite face $ABQH$ is $[\delta u\ \Delta y\ \Delta z]_{(x+\Delta x,\ y_4,\ z_4)}$ nearly. Hence the rate at which the amount of fluid in the element is changing due to these two faces is given approximately by

$$[\delta u\ \Delta y\ \Delta z]_{(x,\ y_4,\ z_4)} - [\delta u\ \Delta y\ \Delta z]_{(x+\Delta x,\ y_4,\ z_4)}$$
$$= -\ \Delta x\ \Delta y\ \Delta z\ \frac{[\delta u]_{(x+\Delta x,\ y_4,\ z_4)} - [\delta u]_{(x,\ y_4,\ z_4)}}{\Delta x}$$

Likewise, the rates of change due to the other two pairs of faces are

$$-\ \Delta x\ \Delta y\ \Delta z\ \frac{[\delta v]_{(x_5,\ y+\Delta y,\ z_5)} - [\delta v]_{(x_5,\ y,\ z_5)}}{\Delta y}$$

and

$$-\ \Delta x\ \Delta y\ \Delta z\ \frac{[\delta w]_{(x_6,\ y_6,\ z+\Delta z)} - [\delta w]_{(x_6,\ y_6,\ z)}}{\Delta z}.$$

As before, each bracket is evaluated at a suitable interior point. Now the weight of the element is, approximately, $\delta\ \Delta x\ \Delta y\ \Delta z$ (lb.), and its time rate of change is $(\partial\delta/\partial t)\ \Delta x\ \Delta y\ \Delta z$. Equating this to the sum of the preceding three expressions, dividing by $\Delta x\ \Delta y\ \Delta z$, and allowing Δx, Δy, Δz all to approach zero, we get in the limit

$$\frac{\partial \delta}{\partial t} + \frac{\partial(\delta u)}{\partial x} + \frac{\partial(\delta v)}{\partial y} + \frac{\partial(\delta w)}{\partial z} = 0. \tag{9}$$

This is the *equation of continuity* for three-dimensional flow; the corresponding equation for two- or for one-dimensional flow is readily deduced. If the fluid is incompressible, so that δ is constant, (9) reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{10}$$

The quantity whose x-, y-, and z-components are

$$\frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \qquad \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \qquad \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{11}$$

is called the *rotation* of the fluid; a small spherical mass of the fluid has an instantaneous angular velocity having these expressions as components. If all three of the expressions (11) vanish identically, the fluid motion is said to be *irrotational*. Now the necessary and sufficient condition that $u\,dx + v\,dy + w\,dz$ be the exact differential of some function of x, y, and z is that the quantities (11) vanish identically.* Hence, if the fluid motion is irrotational, there exists a function, which we denote by $-\phi(x, y, z)$, such that

$$-d\phi = u\,dx + v\,dy + w\,dz. \tag{12}$$

The function ϕ so determined is called the *velocity potential*. Since $d\phi = \phi_x\,dx + \phi_y\,dy + \phi_z\,dz$, we have from (12),

$$u = -\frac{\partial \phi}{\partial x}, \qquad v = -\frac{\partial \phi}{\partial y}, \qquad w = -\frac{\partial \phi}{\partial z}. \tag{13}$$

When the motion is irrotational and the fluid incompressible, equations (9) and (13) together give us

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0. \tag{14}$$

Consequently the velocity potential satisfies Laplace's equation (equation (8), Art. 32).

The *streamlines* of fluid motion at any instant are defined as curves such that the tangent at any point of each gives the direction of flow at that point and at that time. Hence the values of the velocity components u, v, w at a point are direction numbers of the tangent to the streamline through that point. Now at any point of a surface of the family $\phi(x, y, z) = \text{const.}$, the partial derivatives ϕ_x, ϕ_y, ϕ_z are direction numbers of the normal to the surface (Art. 21). It therefore follows from relations (13) that the streamlines cut all of the velocity equipotential surfaces, $\phi(x, y, z) = \text{const.}$, orthogonally.

* This can be deduced from equations (4)–(6) of Art. 19, with $\lambda = 1$.

Suppose now that the motion of an incompressible homogeneous fluid is irrotational, and that no external forces are acting. From the first supposition, (11) and (13) give us

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}, \qquad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \qquad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y},$$

$$u = -\frac{\partial \phi}{\partial x}, \qquad v = -\frac{\partial \phi}{\partial y}, \qquad w = -\frac{\partial \phi}{\partial z}.$$

Using these relations, and setting $G_1 = G_2 = G_3 = 0$, equations (3)–(5) become

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x} - \frac{\partial}{\partial x}\frac{\partial \phi}{\partial t} &= -\frac{g}{\delta}\frac{\partial F}{\partial x}, \\ u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial y} - \frac{\partial}{\partial y}\frac{\partial \phi}{\partial t} &= -\frac{g}{\delta}\frac{\partial F}{\partial y}, \qquad (15) \\ u\frac{\partial u}{\partial z} + v\frac{\partial v}{\partial z} + w\frac{\partial w}{\partial z} - \frac{\partial}{\partial z}\frac{\partial \phi}{\partial t} &= -\frac{g}{\delta}\frac{\partial F}{\partial z}. \end{aligned}$$

Let V denote the resultant velocity: $V^2 = u^2 + v^2 + w^2$; then equations (15) may be written in the form

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{V^2}{2}\right) - \frac{\partial}{\partial x}\frac{\partial \phi}{\partial t} &= -\frac{g}{\delta}\frac{\partial F}{\partial x}, \\ \frac{\partial}{\partial y}\left(\frac{V^2}{2}\right) - \frac{\partial}{\partial y}\frac{\partial \phi}{\partial t} &= -\frac{g}{\delta}\frac{\partial F}{\partial y}, \\ \frac{\partial}{\partial z}\left(\frac{V^2}{2}\right) - \frac{\partial}{\partial z}\frac{\partial \phi}{\partial t} &= -\frac{g}{\delta}\frac{\partial F}{\partial z}. \end{aligned}$$

Multiplying respectively by dx, dy, and dz, and adding, we get for any fixed value of t,

$$d\left(\frac{V^2}{2}\right) - d\left(\frac{\partial \phi}{\partial t}\right) = -\frac{g}{\delta}\,dF.$$

Integration then gives us

$$\frac{V^2}{2} - \frac{\partial \phi}{\partial t} = -\frac{g}{\delta}F + f(t), \qquad (16)$$

where f is an arbitrary function of t. If the motion is steady, so that F, V, and ϕ are independent of time, $f(t)$ will be merely a constant, whence

$$F = -\frac{\delta}{2g} V^2 + \text{const.} \tag{17}$$

EXERCISES

1. Consider two-dimensional steady irrotational motion of an incompressible fluid near two bounding walls at right angles to each other. It may then be shown that the streamlines are rectangular hyperbolas, $xy = \text{const.}$, having the walls as asymptotes. Show that a permissible potential function is $\phi = x^2 - y^2$ by showing that ϕ satisfies Laplace's equation and by proving that the curves $x^2 - y^2 = \text{const.}$ are orthogonal trajectories of the family $xy = \text{const.}$ Hence show that the pressure at any point is

$$F(x, y) = F(0, 0) - \frac{2\delta}{g}(x^2 + y^2).$$

2. If the walls in Exercise 1 make an angle of 60° with each other, the streamlines are the curves $3x^2y - y^3 = \text{const.}$ Show that a permissible potential function is $\phi = x^3 - 3xy^2$, and that the corresponding pressure is

$$F(x, y) = F(0, 0) - \frac{9\delta}{2g}(x^2 + y^2)^2.$$

3. The streamlines of steady irrotational two-dimensional flow of an incompressible fluid about a rotating circular cylinder are circles, $x^2 + y^2 = \text{const.}$ Show that a permissible potential function is $\phi = -\arctan(y/x)$, and that the corresponding pressure is

$$F(x, y) = -\frac{\delta}{2g(x^2 + y^2)},$$

assuming zero pressure at an infinite distance from the cylinder.

4. Consider two-dimensional steady flow of an incompressible fluid, in which no external forces act, but for which the rotation $\omega = \frac{1}{2}(v_x - u_y) \neq 0$. Show that

$$u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = 0,$$

and hence that the rotation is constant.

5. In a certain steady flow of an incompressible fluid, with no external forces acting, the components of velocity are the same as those of a particle moving with constant angular velocity k in a counterclockwise direction along the circle $x^2 + y^2 = a^2$. Show that the rotation of the fluid is equal to the angular velocity of the particle.

CHAPTER IV

LINEAR EQUATIONS OF FIRST ORDER

Partial differential equations of the first order, that is, equations containing one or more partial derivatives of only the first order, may be divided into two broad classes: linear equations and non-linear equations. By a linear equation of the first order is meant an equation which is of the first degree in the partial derivatives; and a non-linear first order equation is an equation in which at least one of the partial derivatives occurs in some way other than to the first degree.

It should be particularly noticed that the above definition of linearity differs from that used in connection with ordinary differential equations. In order that an ordinary equation be linear, it is necessary that the dependent variable as well as its derivatives be present only to the first degree (Arts. 4, 10). In linear first order partial equations, however, the dependent variable may occur in any manner, so long as the equation is linear in the derivatives involved in it.

It is our purpose in this chapter to discuss linear first order equations. Non-linear equations of the first order will be considered in Chapter V.

35. Subsidiary equations. The type form of a linear partial differential equation of the first order and involving two independent variables x and y and the dependent variable z is

$$Pp + Qq = R, \tag{1}$$

where P, Q, R are functions of x, y, and z. As usual, p and q, respectively, denote $\partial z/\partial x$ and $\partial z/\partial y$. Equation (1) is called *Lagrange's linear equation.*

Let $z = f(x, y)$ be a particular solution of equation (1). Considering a fixed point (x, y, z) on the surface $z = f(x, y)$, we can give a simple geometric meaning to equation (1). Since p, q, -1 are direction numbers of the normal N to the surface at (x, y, z), (1) tells us that this normal is perpendicular to a line L through (x, y, z) and with direction numbers P, Q, R (Arts. 21–22).

Now let the plane of N and L cut the surface in the curve C, having direction numbers dx, dy, dz (Art. 23). Since the curve C and the line L

have the same direction, the two sets of direction numbers are proportional:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{2}$$

The simultaneous ordinary differential equations (2) are called the *subsidiary equations* for Lagrange's equation (1).

In the light of the foregoing geometric argument, we should expect a close connection between the solutions of the subsidiary equations (2) and the integrals of the Lagrange equation (1). *Lagrange's method*, to be considered in the next article, will serve to bring out this relationship.

36. The general integral. Let $u(x, y, z) = a$ and $v(x, y, z) = b$, where u and v are definite functions of x, y, and z, and a and b are arbitrary constants, be two independent solutions of the ordinary differential equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{1}$$

We assume that both u and v involve z explicitly. For convenience, let $d\lambda$ denote the common value of the three ratios in (1), so that $dx = P\,d\lambda$, $dy = Q\,d\lambda$, $dz = R\,d\lambda$.

Now, taking the total differential (Art. 19) of one of the solutions of (1), say $u = a$, we get

$$du = u_x\,dx + u_y\,dy + u_z\,dz = 0.$$

Replacing dx, dy, and dz respectively by $P\,d\lambda$, $Q\,d\lambda$, and $R\,d\lambda$, and cancelling $d\lambda$, we have

$$Pu_x + Qu_y + Ru_z = 0, \tag{2}$$

whence

$$-P\frac{u_x}{u_z} - Q\frac{u_y}{u_z} = R. \tag{3}$$

But, for the surface $u = a$, $-u_x/u_z = p$, $-u_y/u_z = q$ (Art. 20). Hence (3) may be written in the form

$$Pp + Qq = R. \tag{4}$$

Therefore the integral $u(x, y, z) = a$ of the subsidiary equations (1) satisfies the linear partial differential equation (4). Similarly, the solution $v(x, y, z) = b$ of (1) is an integral of (4). Accordingly, *the subsidiary equations* (1) *furnish us with two independent solutions of the partial differential equation* (4).

Now let $\phi(u, v) = 0$ be an arbitrary functional relation between the independent integrals u and v of (1). We have already found (Art. 28, Example 2), that the elimination of the arbitrary function ϕ leads to the equation

$$(u_y v_z - u_z v_y)p + (u_z v_x - u_x v_z)q = u_x v_y - u_y v_x. \tag{5}$$

If we solve the equations

$$\begin{aligned} Pu_x + Qu_y &= -Ru_z, \\ Pv_x + Qv_y &= -Rv_z, \end{aligned} \tag{6}$$

(obtained from (2) and its analogue for the surface $v = b$) for P and Q in terms of R, we get

$$P = \frac{u_y v_z - u_z v_y}{u_x v_y - u_y v_x} R, \qquad Q = \frac{u_z v_x - u_x v_z}{u_x v_y - u_y v_x} R.$$

These relations may be combined to yield

$$\frac{u_y v_z - u_z v_y}{P} = \frac{u_z v_x - u_x v_z}{Q} = \frac{u_x v_y - u_y v_x}{R}. \tag{7}$$

Hence, if μ denotes the common value of the ratios (7), we have

$$u_y v_z - u_z v_y = P\mu, \qquad u_z v_x - u_x v_z = Q\mu, \qquad u_x v_y - u_y v_x = R\mu.$$

Substituting in (5), we therefore get, after cancellation of μ,

$$Pp + Qq = R,$$

which is our linear partial differential equation (4).

Therefore *any arbitrary functional relation* $\phi(u, v) = 0$ *satisfies the linear partial differential equation* (4) *provided that* $u = a$ *and* $v = b$ *are independent integrals of the subsidiary equations* (1). The solution $\phi(u, v) = 0$, with ϕ arbitrary, is called the *general integral* of (4).

It was originally assumed that both u and v actually contain z. This was necessary in order that u_z (and v_z) be not identically zero, so that equation (3) in u could be obtained from (2) (and similarly for v), thereby giving meaning to the statements that $u = a$ and $v = b$ satisfy (4). But since any function of two such expressions containing z is also a solution of (4), it is evident that a u- or v-function lacking z may be taken from the subsidiary equations (1) as one argument of the general integral $\phi(u, v) = 0$.

37. Methods of solution. We turn now to a consideration of several methods of obtaining solutions $u = a$ and $v = b$ of the subsidiary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{1}$$

In many cases, the functions P, Q, R are such that the system (1) is or can be made equivalent to two ordinary differential equations each of which involves only two variables and their differentials. The usual methods (Chapter I) for solving these differential equations may then be applicable.

Example 1. Find the general integral of the Lagrange equation

$$xp + yq = z. \tag{2}$$

Solution. Since $P = x$, $Q = y$, $R = z$, the subsidiary equations are here

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}. \tag{3}$$

In these, the variables are separated (cf. Art. 2), and we easily find solutions. From the equality of the first and second ratios, we get, upon integration, $\log y = \log x + \log a$, or $u \equiv y/x = a$. From the first and third ratios in (3), we also find $\log z = \log x + \log b$, or $v \equiv z/x = b$. Hence the general integral of the given Lagrange equation is

$$\phi\left(\frac{y}{x}, \frac{z}{x}\right) = 0, \tag{4}$$

where ϕ is arbitrary. It will be recalled that, in Example 1 of Art. 28, equation (2) was obtained as the eliminant corresponding to the arbitrary functional relation (4). Here we have found that, conversely, (4) is the general integral of (2).

In some problems it is possible easily to find one integral $u = a$ and then to find a second integral $v = b$ with the aid of the first.

Example 2. Find the general integral of

$$xp - yq = xy. \tag{5}$$

Solution. The subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{xy}. \tag{6}$$

From the first and second of these ratios, we find immediately, $\log x + \log y = \log a$, or $u \equiv xy = a$. If, now, we replace xy in the denominator of the third ratio by a, we get from the first and third ratios, $z = a \log x + b$. Thus, since

$a = xy$, we deduce the integral $v \equiv z - xy \log x = b$. It is easily verified that $z - xy \log x = b$ is an integral of (5). Therefore the general integral of (5) is

$$\phi(xy, z - xy \log x) = 0. \tag{7}$$

Other possible methods of attack arise from the following considerations. Setting

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = d\lambda,$$

we see that for any quantities f, g, h, whether constants or variable functions of x, y, and z, we have

$$\frac{f\,dx + g\,dy + h\,dz}{fP + gQ + hR} = \frac{fP\,d\lambda + gQ\,d\lambda + hR\,d\lambda}{fP + gQ + hR} = d\lambda.$$

Therefore the three equal ratios in the subsidiary equations are also equal to

$$\frac{f\,dx + g\,dy + h\,dz}{fP + gQ + hR} \tag{8}$$

for any choice of f, g, and h. We call f, g, and h *multipliers* for the subsidiary equations.

Now it may be possible to choose two sets of multipliers, f_1, g_1, h_1 and f_2, g_2, h_2, such that

$$\frac{f_1\,dx + g_1\,dy + h_1\,dz}{f_1P + g_1Q + h_1R} = \frac{f_2\,dx + g_2\,dy + h_2\,dz}{f_2P + g_2Q + h_2R}$$

is integrable. This will lead to one of the desired integrals of the subsidiary equations.

Example 3. Find the general integral of

$$(y + z)p + (z + x)q = x + y. \tag{9}$$

Solution. The subsidiary equations are

$$\frac{dx}{y + z} = \frac{dy}{z + x} = \frac{dz}{x + y}. \tag{10}$$

Here several useful sets of multipliers exist. Making use of three sets:

$$1, 1, 1; \qquad 1, -1, 0; \qquad 1, 0, -1;$$

we get as a system equivalent to (10),

$$\frac{dx + dy + dz}{2(x + y + z)} = \frac{dx - dy}{y - x} = \frac{dx - dz}{z - x}.$$

From the first and second of these ratios, we find $\log(x+y+z) + 2\log(x-y) = \log a$, or $u \equiv (x+y+z)(x-y)^2 = a$. From the first and third ratios, we also get $\log(x+y+z) + 2\log(x-z) = \log b$, or $v \equiv (x+y+z)(x-z)^2 = b$. Hence the general integral of (9) is

$$\phi[(x+y+z)(x-y)^2, \quad (x+y+z)(x-z)^2] = 0. \tag{11}$$

It may sometimes be possible so to choose multipliers f, g, h that the expression $fP + gQ + hR$ in the denominator of (8) vanishes identically, while at the same time the numerator $f\,dx + g\,dy + h\,dz$ is the exact differential of some function $u(x, y, z)$. In this event an integral is evidently obtainable, for $f\,dx + g\,dy + h\,dz = du$ must be equal to zero, whence we get $u = a$.

Example 4. Find the general integral of

$$(y-z)p + (z-x)q = x-y. \tag{12}$$

Solution. The subsidiary equations are

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}. \tag{13}$$

With the multipliers 1, 1, 1, we have $1\cdot(y-z) + 1\cdot(z-x) + 1\cdot(x-y) = 0$, and $1\cdot dx + 1\cdot dy + 1\cdot dz = d(x+y+z) = 0$, so that $u \equiv x+y+z = a$. Also, with the multipliers x, y, z, we have $x(y-z) + y(z-x) + z(x-y) = 0$, and $x\,dx + y\,dy + z\,dz = 0$, whence $v \equiv x^2 + y^2 + z^2 = b$. Therefore the general integral of (12) is

$$\phi(x+y+z, x^2+y^2+z^2) = 0. \tag{14}$$

EXERCISES

Find the general integral of each of the following Lagrange equations. A, B, and C denote constants.

1. $2p - 3q = 4$.
2. $Ap + Bq = C$.
3. $xp - yq = z$.
4. $xyp - x^2q + yz = 0$.
5. $yp + xq = 0$.
6. $x^2p + y^2q = z^2$.
7. $(x+A)p + (y+B)q = z + C$.
8. $yp + xq = xy$.
9. $yzp + zxq = xy$.
10. $(x-z)p + (z-y)q = 0$.
11. $(y^2 - z^2)p + (z^2 - x^2)q = x^2 - y^2$.
12. $(x+y)(xp - yq) = (x-y)z$.
13. $p - q = \log(x+y)$.
14. $yzp - xzq = xy(x^2 + y^2)$.
15. $xp + 2yq = (x+y)z$.
16. $x^2p - y^2q = (x-y)z$.
17. $(2z - y)p + (x+z)q + 2x + y = 0$.
18. $x(y-z)p + y(z-x)q = z(x-y)$.
19. $xy(p-q) = (x-y)z$.
20. $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$.

38. Complete integrals. We have seen that Lagrange's method of solving a linear partial differential equation of the first order,

$$Pp + Qq = R, \tag{1}$$

leads directly to the so-called general integral,

$$\phi(u, v) = 0, \tag{2}$$

where ϕ is an arbitrary function of the arguments $u(x, y, z)$ and $v(x, y, z)$, and $u = a$, $v = b$ are two independent integrals of the subsidiary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{3}$$

In some instances, such as geometric problems, we can deal with particular solutions more conveniently than with the general integral. The most important type of particular solution obtainable from the general integral is that containing two arbitrary constants, say α and β. Such a solution of (1), which we denote by

$$f(x, y, z, \alpha, \beta) = 0, \tag{4}$$

is called a *complete integral*.

If $u = a$ and $v = b$ are two independent solutions of the subsidiary equations (3), then

$$v = \alpha u + \beta \tag{5}$$

may be taken as a complete integral. For, since u and v separately satisfy (1) (Art. 36), (5) will be a solution; and since, moreover, (5) contains two arbitrary constants, it is a complete integral.

It is evident that the two-parameter family of surfaces (5) does not possess an envelope (Art. 26), at least as given by this equation, for partial differentiation with respect to β leads to the absurdity $0 = 1$. We shall defer further discussion of this point until the following article.

Suppose now that the two-parameter family of surfaces (5) is made into a one-parameter family by stipulating that β shall be some specified function of α, say $\beta = g(\alpha)$. Then

$$v = \alpha u + g(\alpha). \tag{6}$$

This relation yields a solution of our partial differential equation, and the surfaces (6) will, in general, possess an envelope. If we differentiate (6) partially with respect to α, we get

$$0 = u + g'(\alpha). \tag{7}$$

From (7), α may, usually, be obtained in terms of u, and if this expression is inserted in place of α in (6), we find a relation

$$v = \psi(u). \tag{8}$$

Since (8) is a particular case of the general integral, it will satisfy the Lagrange equation (1).

Being a part of the general integral, the envelope (8) of the surfaces (6) does not, however, furnish us with a new solution. Thus the situation here differs from that encountered in connection with ordinary differential equations. There, the envelope (when it exists) of a one-parameter family of curves yields a singular solution (Art. 7) which is not a part of the general solution.

If the function $g(\alpha)$ in (6) is arbitrary, the derived equation (7) gives us α as an equally arbitrary function of u, whence the ψ-function in (8) is likewise arbitrary. Thus (8), which is then the explicit equivalent of $\phi(u, v) = 0$, represents the general integral. Therefore the general integral may be regarded as the aggregate of the envelopes of all one-parameter families of surfaces obtainable from a complete integral.

Equations (7) and (6) may be written in the form

$$u = -g'(\alpha), \qquad v = -\alpha g'(\alpha) + g(\alpha). \tag{9}$$

These equations, giving u and v as functions of a parameter α, simultaneously represent certain space curves called *characteristic curves* (Art. 26). The envelope (8) is tangent to each surface of the family (6) along a corresponding characteristic curve. It is apparent that relations (9), with $g(\alpha)$ arbitrary, yield precisely the curves given by the subsidiary equations (3).

39. Special integrals. In Lagrange's linear equation,

$$Pp + Qq = R, \tag{1}$$

it is, of course, necessary that not both of the functions P, Q be identically zero. Suppose, for definiteness, that P does not vanish identically,* so that we can write (1) in the form

$$p = -\frac{Q}{P}q + \frac{R}{P}. \tag{2}$$

Let the right member of (2) be analytic in the neighborhood of $x = x_0$, $y = y_0$, $z = z_0$, $q = q_0$; that is, let p be expressible as a power series in

* If P is identically zero, we can formulate a similar statement by interchanging x and y, and p and q.

$x - x_0$, $y - y_0$, $z - z_0$, $q - q_0$, convergent in some region. Further, let $g(y)$ be some function of y analytic in the neighborhood of $y = y_0$, and such that $g(y_0) = z_0$, $g'(y_0) = q_0$. Then it may be proved that there exists a unique solution $z = G(x, y)$, analytic in the neighborhood of $x = x_0$, $y = y_0$, and such that $G(x_0, y) \equiv g(y)$.

When the conditions of the above existence theorem are satisfied, there can be no solutions of (1) other than those given by the general integral. However, when the right member of equation (2) is not analytic in some particular region, solutions not belonging to the general integral may sometimes exist. Such solutions are called *special integrals.*

As an example, consider the Lagrange equation

$$p + q = \sqrt{z}. \tag{3}$$

The function $\sqrt{z} - q$ is analytic in every region for which $z \neq 0$, but $z = 0$ must be excluded since $\sqrt{z}$ cannot be expanded in ascending integral powers of z. By the methods of Art. 37, we readily find, as integrals of the subsidiary equations

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{\sqrt{z}}$$

the solutions $u \equiv y - x = a$, $v \equiv 2\sqrt{z} - x = b$, and consequently the general integral of the given Lagrange equation (3) is

$$\phi(y - x, \quad 2\sqrt{z} - x) = 0. \tag{4}$$

Now it is immediately evident that $z = 0$ is a solution of (3). But this solution cannot be expressed as a function of u and v,* that is, it is not a part of the general integral (4). Hence $z = 0$ is a special integral.

It is often possible to give a geometric meaning to a special integral. For example, consider the complete integral

$$2\sqrt{z} - x = \alpha(y - x) + \beta \tag{5}$$

of equation (3). As has been stated (Art. 38), the two-parameter family of surfaces given by a complete integral in a form such as (5) does not yield an envelope. But let us rationalize (5) to obtain the relation

$$[x + \alpha(y - x) + \beta]^2 = 4z,$$

or

$$x^2 + \alpha^2(y - x)^2 + \beta^2 + 2\alpha x(y - x) + 2\beta x + 2\alpha\beta(y - x) - 4z = 0. \tag{6}$$

* See Exercise 6 below.

For a chosen pair of values of α and β, (6) represents a parabolic cylinder, and the corresponding equation (5) represents half that cylinder. That is, (6) is equivalent to (5) and

$$-2\sqrt{z} - x = \alpha(y - x) + \beta, \tag{5'}$$

(which is a complete integral of $p + q = -\sqrt{z}$) taken together.

The parabolic cylinders (6) can be shown to possess an envelope If we differentiate (6) partially with respect to α, or with respect to β, we get

$$x + \alpha(y - x) + \beta = 0. \tag{7}$$

Combining this equation with (6), we therefore get, as the eliminant,

$$z = 0. \tag{8}$$

This equation, of the xy-plane, represents the envelope of the parabolic cylinders (6), and the special integral of (3).

By a change of variable, we can sometimes exhibit the existence of a special integral. Thus, if we set $\sqrt{z} = w$, so that $z = w^2$, then $p = 2ww_x$, $q = 2ww_y$, and (3) is transformed into

$$2ww_x + 2ww_y = w. \tag{9}$$

Evidently $w = 0$ satisfies (9), so that $z = 0$ is an integral of (3). After removing the factor w from (9), the resulting Lagrange equation leads to the general integral (4) as before.

A Lagrange equation, for which the analyticity conditions of the existence theorem are not fulfilled, need not possess a special integral. A change of variable will sometimes indicate, by the presence of a factor as in the above example, that a relation $w = 0$, as well as the integrals $u = a$ and $v = b$ of the subsidiary equations, satisfies the equation, but $w = 0$ may actually be a part of the general integral (see Exercise 7 below).

EXERCISES

1. In Example 1 of Art. 37, show that the complete integral $v = \alpha u + \beta$ represents a two-parameter family of planes through the origin. Taking $\beta = \alpha^2$, show also that the resulting one-parameter family of planes has a quadric cone as envelope, and that this cone is part of the general integral. What are the characteristic curves in this case?

2. In Example 4 of Art. 37, show that the complete integral $v = \alpha u + \beta$ represents a two-parameter family of spheres, all with centers on a certain line. By suitably choosing β as a function of α, obtain the equation of the one-parameter family of spheres tangent to the coordinate planes, and show that these spheres have a quadric cone as envelope. What are the characteristic curves?

3. Show that the Lagrange equation $yp - xq = 0$ possesses the complete integral $z = \alpha(x^2 + y^2) + \beta$, a family of paraboloids of revolution. Find the envelope of the one-parameter family obtained by taking $\beta = \alpha^2$, and show that the envelope is a surface of revolution and a part of the general integral.

4. Show that the Lagrange equation $zq + y = 0$ has the complete integral $x^2 + y^2 + z^2 = \alpha x + \beta$, a family of spheres. Find the envelope of the one-parameter family obtained by taking $\beta = 1 - \alpha^2/4$, and show that the envelope is a part of the general integral.

5. Show that the Lagrange equation $(2z - y)p + (x + z)q + 2x + y = 0$ has the complete integral $x^2 + y^2 + z^2 = \alpha(x - 2y - z) + \beta$, a family of spheres. Find the envelope of the one-parameter family obtained by setting $\beta = 1 - 3\alpha^2/2$, and show that the envelope is a part of the general integral.

6. Show that $w \equiv z$ cannot be expressed as a function of $u \equiv y - x$ and $v \equiv 2\sqrt{z} - x$ by showing that the Jacobian $\partial(u, v, w)/\partial(x, y, z)$ is not identically zero (Art. 24), whence $z = 0$ is a special integral of equation (3), Art. 39.

7. Show that the Lagrange equation $\sqrt{z - y}\ p + q = 1$ may be transformed into $w(ww_x + w_y) = 0$, where $w = \sqrt{z - y}$, but that $w = 0$, or $z = y$, is not a special integral.

8. Find the general integral and the special integral of the Lagrange equation $(1 + \sqrt{z - x})p - q = 1$.

9. Find the general integral and two special integrals of the Lagrange equation $\sqrt{z - x}\ p + \sqrt{z - y}\ q = 0$.

10. Find the general integral and the special integral of the Lagrange equation $\sqrt{x}\ p - q = \sqrt{z + x} - \sqrt{x}$.

40. Linear equations with n independent variables. The linear partial differential equation of first order in the n independent variables $x_1, x_2, \cdots, x_n$ and the dependent variable z is of the form

$$P_1p_1 + P_2p_2 + \cdots + P_np_n = R, \tag{1}$$

where $p_j = \partial z/\partial x_j$ $(j = 1, 2, \cdots, n)$, and the P's and R are functions of the x's and z. When $n = 2$, equation (1) is evidently Lagrange's equation.

The system of n independent ordinary differential equations

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \cdots = \frac{dx_n}{P_n} = \frac{dz}{R} \tag{2}$$

are known as the equations subsidiary to equation (1). These subsidiary equations have, we suppose, n independent integrals $u_j(x_1, \cdots, x_n, z) = a_j$ $(j = 1, 2, \cdots, n)$, where the a's are arbitrary constants.

When $n > 2$, it may be shown by the same method employed for the Lagrange equation that

$$\phi(u_1, u_2, \cdots, u_n) = 0, \tag{3}$$

where ϕ is an arbitrary function of its n arguments, is a solution of equation (1). Relation (3) is called the general integral of the linear partial differential equation (1). A relation involving n arbitrary constants and satisfying (1) is called a complete integral of (1), and in some cases an equation (1) with $n > 2$ may possess special integrals.

To solve the subsidiary equations (2), and thus to solve a linear equation (1), when $n > 2$, the methods of Art. 37 may also be used.

Example. Find the general integral of the linear equation

$$x_1p_1 + x_2p_2 + \cdots + x_np_n = -z.$$

Solution. Here the subsidiary equations are

$$\frac{dx_1}{x_1} = \frac{dx_2}{x_2} = \cdots = \frac{dx_n}{x_n} = -\frac{dz}{z}.$$

It is immediately seen that we can take as n independent solutions the relations

$$u_1 \equiv x_1z = a_1, \quad u_2 \equiv x_2z = a_2, \quad \cdots, \quad u_n \equiv x_nz = a_n,$$

and consequently the general integral of the given equation is given by the relation

$$\phi(x_1z, x_2z, \cdots, x_nz) = 0.$$

When $n > 2$, equation (1) and its solutions do not have geometric interpretations as do the Lagrange equation and its solutions. However, it is customary to speak of (1) as $(n + 1)$-dimensional.

41. Homogeneous equations lacking the dependent variable. A homogeneous linear partial differential equation of the first order, with coefficients free of the dependent variable F, is an equation of the form

$$Q_1\frac{\partial F}{\partial x_1} + Q_2\frac{\partial F}{\partial x_2} + \cdots + Q_n\frac{\partial F}{\partial x_n} = 0, \tag{1}$$

where $Q_1, Q_2, \cdots, Q_n$ are functions of the n independent variables $x_1, x_2, \cdots, x_n$ but do not involve F.

The subsidiary equations belonging to (1) may be written as

$$\frac{dx_1}{Q_1} = \frac{dx_2}{Q_2} = \cdots = \frac{dx_n}{Q_n}, \quad dF = 0. \tag{2}$$

If $u_j(x_1, x_2, \cdots, x_n) = a_j$ $(j = 1, 2, \cdots, n - 1)$, involving only the x's, are $n - 1$ independent integrals of (2), then the general integral of (1) is given by

$$F = \psi(u_1, u_2, \cdots, u_{n-1}), \tag{3}$$

where ψ is arbitrary.

The homogeneous equation (1) may thus be regarded as merely a special type of $(n + 1)$-dimensional equation. But since the variable F is lacking, it may also be viewed as equivalent to an n-dimensional equation. For, let $v(x_1, x_2, \cdots, x_n) = 0$ be an integral of (1), so that

$$Q_1 \frac{\partial v}{\partial x_1} + Q_2 \frac{\partial v}{\partial x_2} + \cdots + Q_n \frac{\partial v}{\partial x_n} = 0 \tag{4}$$

identically. Now consider $v = 0$ as an implicit relation defining, say, x_n in terms of the remaining $n - 1$ x's, so that (Art. 20)

$$\frac{\partial x_n}{\partial x_j} = -\frac{\dfrac{\partial v}{\partial x_j}}{\dfrac{\partial v}{\partial x_n}} \quad (j = 1, 2, \cdots, n - 1).$$

Then (4) becomes

$$Q_1 \frac{\partial x_n}{\partial x_1} + Q_2 \frac{\partial x_n}{\partial x_2} + \cdots + Q_{n-1} \frac{\partial x_n}{\partial x_{n-1}} = Q_n, \tag{5}$$

which is in general a non-homogeneous n-dimensional equation of the type considered in Art. 40.

Example. Consider the linear equation

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \sqrt{z} \frac{\partial F}{\partial z} = 0.$$

Regarded as four-dimensional, this has the subsidiary equations

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{\sqrt{z}}, \qquad dF = 0,$$

whence we easily get

$$F = \psi(y - x, 2\sqrt{z} - x)$$

as the general integral. On the other hand, we may regard the given equation as equivalent to the three-dimensional equation

$$p + q = \sqrt{z},$$

which was given as an example in Art. 39, and there was found to have as general integral,

$$\phi(y - x, 2\sqrt{z} - x) = 0.$$

In Art. 39 it was further found that $z = 0$ is a special integral of $p + q = \sqrt{z}$. But with $F = z$, we get

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \sqrt{z}\,\frac{\partial F}{\partial z} = \sqrt{z},$$

which does not vanish identically. Hence a solution which serves as a special integral of the equivalent three-dimensional equation is not a solution of any kind for the four-dimensional homogeneous equation.

The non-existence of a special integral in the above example of a homogeneous equation is an instance of the following:

THEOREM. *The homogeneous equation lacking the dependent variable,*

$$Q_1 \frac{\partial F}{\partial x_1} + Q_2 \frac{\partial F}{\partial x_2} + \cdots + Q_n \frac{\partial F}{\partial x_n} = 0, \tag{1}$$

regarded as $(n + 1)$-dimensional, has no special integrals.

To prove this theorem, let $u_j(x_1, x_2, \cdots, x_n) = a_j$ $(j = 1, 2, \cdots, n - 1)$, be $n - 1$ independent integrals of the subsidiary equations (2). Then we have

$$Q_1 \frac{\partial u_j}{\partial x_1} + Q_2 \frac{\partial u_j}{\partial x_2} + \cdots + Q_n \frac{\partial u_j}{\partial x_n} = 0 \tag{6}$$

$(j = 1, 2, \cdots, n - 1)$. Now if $F = f(x_1, x_2, \cdots, x_n)$ is any integral of (1), we also have

$$Q_1 \frac{\partial f}{\partial x_1} + Q_2 \frac{\partial f}{\partial x_2} + \cdots + Q_n \frac{\partial f}{\partial x_n} = 0. \tag{7}$$

The $n - 1$ equations (6) are satisfied identically, and not by virtue of the relations $u_j = a_j$, for the a's do not appear in (6). Likewise, (7) is an identity, since f does not appear in it. The system of n homogeneous algebraic equations (6)–(7) in the Q's has a non-trivial set of solutions; therefore the determinant of the coefficients must vanish identically. But this determinant is the Jacobian

$$\frac{\partial(u_1, u_2, \cdots, u_{n-1}, f)}{\partial(x_1, x_2, \cdots, x_n)},$$

and its identical vanishing tells us that the u's and f are functionally dependent (Art. 24). Therefore f, which is any solution of (1), is expressible in terms of the u's,

$$f = \psi_1(u_1, u_2, \cdots, u_{n-1});$$

that is, f is a part of the general integral. Hence (1) possesses no special integrals.

EXERCISES

In Exercises 1–12, find the general integral of each linear equation, in which p_i denotes $\partial z/\partial x_i$.

1. $p_1 + p_2 + p_3 = 1$.
2. $p_1 - p_2 + p_3 - p_4 = 0$.
3. $x_1p_1 + x_2p_2 + x_3p_3 = z$.
4. $x_1p_1 - x_2p_2 + x_3p_3 + z = 0$.
5. $x_2p_1 + x_1p_2 + x_1x_2p_3 = 0$.
6. $x_1x_3p_1 - x_2x_3p_2 + p_3 = z$.
7. $z(p_1 + p_2 + p_3) = 1$.
8. $z(x_1p_1 + x_2p_2 + x_3p_3) = 1$.
9. $x_2x_3p_1 + x_1x_3p_2 + x_1x_2p_3 = 0$.
10. $z(x_2x_3p_1 + x_1x_3p_2 + x_1x_2p_3) = x_1x_2x_3$.
11. $x_1p_1 - x_2p_2 + x_1x_2x_3p_3 = z$.
12. $(x_2 - x_3)p_1 + (x_3 - x_1)p_2 + (x_1 - x_2)p_3 = 0$.

13. Construct a four-dimensional equation equivalent to the five-dimensional homogeneous equation of Exercise 2, in which z is lacking, and find the general solution of the equivalent equation.

14. Construct a three-dimensional equation equivalent to the four-dimensional homogeneous equation of Exercise 5, and find the general solution of the equivalent equation.

15. Construct a three-dimensional equation equivalent to the four-dimensional homogeneous equation of Exercise 9, and find the general solution of the equivalent equation.

16. Construct a three-dimensional equation equivalent to the four-dimensional homogeneous equation of Exercise 12, and find the general solution of the equivalent equation.

17. Construct the four-dimensional homogeneous equation equivalent to the three-dimensional equation of Exercise 8, Art. 39, and show that the special integral of that exercise does not satisfy the equivalent homogeneous equation.

18. Construct the four-dimensional homogeneous equation equivalent to the equation of Exercise 9, Art. 39, and show that the special integrals of that exercise do not satisfy the equivalent homogeneous equation.

19. Construct the four-dimensional homogeneous equation equivalent to the equation of Exercise 10, Art. 39, and show that the special integral of that exercise does not satisfy the equivalent homogeneous equation.

20. Show that the four-dimensional homogeneous equation

$$\sqrt{x_1}\,\frac{\partial F}{\partial x_1} + \sqrt{x_2}\,\frac{\partial F}{\partial x_2} + \sqrt{x_3}\,\frac{\partial F}{\partial x_3} = 0$$

has $F = \psi(\sqrt{x_2} - \sqrt{x_1}, \sqrt{x_3} - \sqrt{x_1})$ as general integral. Regarding x_1, x_2, and x_3 in turn as dependent variable, construct three equivalent three-dimensional equations, and show that each has a special integral which does not satisfy the given homogeneous equation.

42. Geometric applications. In Art. 29 we indicated some of the ways in which a geometric problem in three dimensions may lead to a partial differential equation. When the partial differential equation so obtained is linear, the methods of this chapter may then be applied to find the surface or surfaces satisfying the stated geometric conditions. We illustrate the process by means of an example.

Example 1. Find the surfaces whose tangent planes all pass through the origin.

Solution. The equation of a plane tangent to a surface at a point (x, y, z) is (Art. 21)

$$p(X - x) + q(Y - y) - (Z - z) = 0, \tag{1}$$

where $p \equiv \partial z/\partial x$ and $q \equiv \partial z/\partial y$ are supposed evaluated at the point (x, y, z). Since this plane is to pass through the origin, we set $X = Y = Z = 0$, and (1) reduces to

$$xp + yq = z. \tag{2}$$

This Lagrange equation must therefore be satisfied by the desired surfaces. We have already solved equation (2) (Example 1, Art. 37), and found as its general integral

$$\phi\left(\frac{y}{x}, \frac{z}{x}\right) = 0. \tag{3}$$

The surfaces (3), which are cones with vertices at the origin (or planes through the origin) for every choice of the function ϕ, are thus all such that their tangent planes pass through the origin.

We consider next the geometric problem of finding a surface satisfying a given Lagrange equation and passing through a given curve. As usual, let $\phi(u, v) = 0$ be the general integral of the given partial differential equation, where $u(x, y, z) = a$ and $v(x, y, z) = b$ are independent integrals of the subsidiary equations, and let the given curve be represented by the equations

$$F(x, y, z) = 0, \qquad G(x, y, z) = 0. \tag{4}$$

Since the coordinates (x, y, z) of any point on the curve (4) are to satisfy the equation of the required surface, we eliminate x, y, and z from the four equations $u = a$, $v = b$, $F = 0$, and $G = 0$ to obtain the relation which must exist between a and b. This in turn implies the necessary functional relation between u and v; that is, it determines the form of the integral $\phi(u, v) = 0$.

Of course, it will not always be possible to perform the necessary elimination. When the elimination process leads to a contradiction, it is an indication that no surface included in the general integral can pass through the given curve.

Example 2. Find the equation of the cone satisfying the Lagrange equation (2) and passing through the circle $x^2 + y^2 + z^2 = 1, x + y + z = 1$.

Solution. We take the general integral of (2) in the form (3), with $u = y/x = a, v = z/x = b$. Then $y = ax, z = bx$, and substitution in the equations of the circle gives us

$$x^2 + y^2 + z^2 = x^2(1 + a^2 + b^2) = 1,$$

$$x + y + z = x(1 + a + b) = 1.$$

Eliminating x between these last equations, we have

$$\frac{1}{x^2} = 1 + a^2 + b^2 = (1 + a + b)^2,$$

$$1 + a^2 + b^2 = 1 + a^2 + b^2 + 2a + 2b + 2ab,$$

$$a + b + ab = 0. \tag{5}$$

Equation (5) is the resulting eliminant. If we now replace a by y/x and b by z/x, we get the required surface,

$$xy + xz + yz = 0. \tag{6}$$

Evidently (6) is a quadric cone fulfilling both conditions.

EXERCISES

1. Find the surfaces whose tangent planes all pass through the point (x_0, y_0, z_0).

2. Find the surfaces such that for each the intercept of the tangent plane on the z-axis is equal to the z-coordinate of the point of tangency.

3. Find the surfaces such that for each the intercept of the tangent plane on the z-axis is equal to the x-coordinate of the point of tangency.

4. Find the surfaces such that for each the intercept of the tangent plane on the y-axis is equal to the square of the z-coordinate of the point of tangency.

5. Find the surfaces all of whose normals pass through the x-axis.

6. Find the most inclusive solution of the Lagrange equation $yzp + zxq + xy = 0$ representing quadric surfaces.

7. Find the most inclusive solution of the Lagrange equation $(z - y)p + (x - z)q = y - x$ representing spheres. What property have these spheres in common?

8. Find the most inclusive solution of the Lagrange equation $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$ representing spheres. What property have these spheres in common?

9. Find the equation of the surface satisfying the Lagrange equation $xp - yq = z$ and passing through the circle $x^2 + y^2 = 1, z = 1$.

10. Find the equation of the surface satisfying the Lagrange equation $xy(p - q) = (x - y)z$ and passing through the hyperbola $y^2 + z^2 = x^2, z = 1$.

43. Physical applications. In the preceding chapter we considered some physical problems yielding partial differential equations. In most cases, we were thereby led to differential equations of order higher than the first, or to systems of first order equations involving more than one dependent variable. To discuss these more fully will, in general, require the methods of later chapters, but we may examine at this time a few elementary matters.

For radial heat flow through a long cylindrical pipe (Art. 31, Exercise 4), we have

$$\frac{\partial Q}{\partial t} = -2\pi K L r \frac{\partial \tau}{\partial r}, \tag{1}$$

where $Q(r, t)$ (cal.) is the quantity of heat flowing through a section at a distance r (cm.) from the axis of the pipe of length L (cm.) and at time t (sec.), K (cal./cm. deg. sec.) is the thermal conductivity of the material of the pipe, and $\tau(r, t)$ (° C.) is the temperature. If the quantity of heat Q is prescribed as a function of r and t, and suitable initial and boundary conditions are also given, equation (1) serves to determine the corresponding temperature τ; or, if τ is preassigned, equation (1) will yield Q.

Similarly, for radial heat flow through a spherical shell (Art. 31, Exercise 5), we have the first order equation

$$\frac{\partial Q}{\partial t} = -4\pi K r^2 \frac{\partial \tau}{\partial r}. \tag{2}$$

In the case of one-dimensional fluid flow (Art. 34), we have the equation of motion,

$$u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = gG - \frac{g}{\delta}\frac{\partial F}{\partial x}, \tag{3}$$

together with the equation of continuity,

$$\frac{\partial \delta}{\partial t} + \frac{\partial (\delta u)}{\partial x} = 0. \tag{4}$$

Here $u(x, t)$ (ft./sec.) is the velocity at a distance x (ft.) from some reference point and at time t (sec.), $G(x, t)$ is the external force (lb.) per unit weight (lb.) of fluid, $\delta(x, t)$ (lb./ft.^{3}) is the density, $F(x, t)$ (lb./ft.^{2}) is the pressure, and $g = 32.2$ ft./sec.^{2} If, in particular, the pressure F does not vary with distance x, equation (3) gives us the velocity u, and when u is known, the density δ can be found from (4). Again, suitable initial and boundary conditions must be given for the complete determination of u and δ as functions of x and t.

Example. Consider one-dimensional fluid flow in which the pressure is independent of distance x (ft.) and for which no external force acts. If the initial velocity at any distance x is numerically equal to $x + 10$ (ft./sec.), and if the initial density is also $x + 10$ (lb./ft.^{3}), find the velocity u and density δ as functions of x and t.

Solution. From equation (3), with $G = 0$, $F_x = 0$, we get

$$u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0, \tag{5}$$

for which the subsidiary equations are

$$\frac{dx}{u} = \frac{dt}{1}, \qquad du = 0.$$

The general integral of (5) will evidently be given by

$$u = \psi_1(x - ut), \tag{6}$$

where ψ_1 is an arbitrary function. Using the initial condition $u(x, 0) = x + 10$, (6) becomes

$$x + 10 = \psi_1(x).$$

Therefore the ψ_1-function of its argument must be such that it is identically equal to that argument increased by 10. Hence the particular solution of (5), meeting the given initial condition, is

$$u = (x - ut) + 10,$$

whence

$$u = \frac{x + 10}{t + 1}. \tag{7}$$

This gives us the velocity at any distance x from our reference point $x = 0$ and at any time $t > 0$.

Now, taking the indicated partial derivative with respect to x of the product δu in (4), and inserting the expression (7) for u, we get

$$\frac{\partial \delta}{\partial t} + \delta \frac{1}{t + 1} + \frac{x + 10}{t + 1} \frac{\partial \delta}{\partial x} = 0,$$

or

$$(x + 10) \frac{\partial \delta}{\partial x} + (t + 1) \frac{\partial \delta}{\partial t} = -\delta. \tag{8}$$

This Lagrange equation has as subsidiary equations,

$$\frac{dx}{x + 10} = \frac{dt}{t + 1} = \frac{d\delta}{-\delta},$$

from which we find as the general integral of (8),

$$(x + 10)\delta = \psi_2[(t + 1)\delta], \tag{9}$$

where ψ_2 is arbitrary. Since we are to have $\delta(x, 0) = x + 10$, (9) yields the relation

$$(x + 10)^2 = \psi_2(x + 10);$$

that is, the ψ_2-function of its argument must be equal to the square of that argument. Consequently the particular solution of (8), fulfilling the stated initial condition, is

$$(x + 10)\delta = (t + 1)^2\delta^2,$$

and therefore the density at distance x and at time t is given by

$$\delta = \frac{x + 10}{(t + 1)^2}. \tag{10}$$

EXERCISES

1. If the quantity of heat Q in a cylindrical pipe or in a spherical shell is independent of distance r and time t, show that the temperature τ is likewise independent of r and t.

2. A cylindrical pipe of inner radius r_1 (cm.) and length L (cm.) contains steam at a temperature τ_1 (° C.). If the quantity of heat is given by $Q = t^2 + 50$ (cal.), where t (sec.) is time, find the temperature τ as a function of r and t.

3. If, in Exercise 2, $Q = 10r \sin t + 50$, find τ.

4. If, in Exercise 2, $Q = 40r^2(1 - e^{-t})$, find τ.

5. A spherical shell of inner radius r_1 (cm.) contains steam at a temperature τ_1 (° C.). If the quantity of heat is given by $Q = 2t^2 + 30$ (cal.), where t (sec.) is time, find the temperature τ as a function of r and t.

6. If, in Exercise 5, $Q = (80 - 20 \cos t)r^2$, find τ.

7. If, in Exercise 5, $Q = 20r(3 - e^{-2t})$, find τ.

8. If, in the example of Art. 43, the initial conditions are $u(x, 0) = 10$ (ft./sec.), $\delta(x, 0) = 64$ (lb./ft.^{3}), find u and δ at distance x and time t.

9. Find the velocity u and density δ for one-dimensional fluid flow in which the pressure is independent of distance x, a constant force $G = 1$ lb. per lb. acts, and the initial conditions are $u(x, 0) = 10$ (ft./sec.), $\delta(x, 0) = 64$ (lb./ft.^{3}).

10. If, in Exercise 9, $G = 6t$, $u(x, 0) = x + 10$, and $\delta(x, 0) = x + 10$, find u and δ.

CHAPTER V

NON-LINEAR EQUATIONS OF FIRST ORDER

In the introduction to the preceding chapter, we defined a non-linear partial differential equation of the first order as a first-order equation in which at least one of the partial derivatives occurs in some way other than to the first degree.

We shall begin our study of such equations by defining the various important types of integrals possessed by them. After this we shall consider the general method, due to Charpit, for solving non-linear equations involving two independent variables, and shall then discuss special types of non-linear equations that can be solved by quite simple methods. Jacobi's method of solving a partial differential equation containing three or more independent variables and the first derivatives of the dependent variable, and systems of such equations, are next considered, and at the end of the chapter we shall investigate some geometric problems depending upon non-linear equations and their solutions.

44. Complete, singular, and general integrals. Let a non-linear partial differential equation of the first order, involving two independent variables x and y and the dependent variable z, be denoted by

$$F(x, y, z, p, q) = 0, \tag{1}$$

where $p \equiv \partial z/\partial x$, $q \equiv \partial z/\partial y$.

A *complete integral* of equation (1) is any solution containing two arbitrary constants, say α and β, and which we denote by

$$f(x, y, z, \alpha, \beta) = 0. \tag{2}$$

This definition of a complete integral is entirely similar to that given for linear equations (Art. 38). Evidently the geometric interpretation of a complete integral is that of a two-parameter family of surfaces.

In general, the arbitrary constants α and β in (2) will not occur linearly, as they do in the complete integral of a Lagrange equation. Accordingly, the two-parameter family of surfaces (2) may possess an envelope, obtained by eliminating α and β from the three relations

$$f = 0, \qquad \frac{\partial f}{\partial \alpha} = 0, \qquad \frac{\partial f}{\partial \beta} = 0. \tag{3}$$

If the eliminant,

$$\psi(x, y, z) = 0, \tag{4}$$

satisfies equation (1), it is said to be a *singular integral* of (1). Since not every eliminant (4) of the equations (3) is an envelope (Art. 26), it is important to test (4) as a possible solution of differential equation (1). Usually a singular integral is distinct from any solution obtainable from a complete integral by assigning particular values to α and β, but this is not always true (see Exercises 3 and 4, Art. 53). A singular integral, when it exists, may also be found by eliminating p and q from the three equations

$$F = 0, \qquad \frac{\partial F}{\partial p} = 0, \qquad \frac{\partial F}{\partial q} = 0. \tag{5}$$

If the arbitrary constants α and β in (2) are not left independent, but are connected by a functional relation,

$$\beta = \phi(\alpha), \tag{6}$$

equation (2) represents a one-parameter family of surfaces,

$$f[x, y, z, \alpha, \phi(\alpha)] = 0. \tag{7}$$

For each choice of the function ϕ, we get, in general, a distinct family, the envelope of which (when it exists) is found by eliminating α between (7) and the equation obtained by differentiating (7) partially with respect to α (Art. 26). The totality of all such envelopes, derived from the equations

$$f[x, y, z, \alpha, \phi(\alpha)] = 0, \qquad \frac{\partial f}{\partial \alpha} = 0, \tag{8}$$

for all possible choices of ϕ, is known as the *general integral* of the partial differential equation (1). In other words, the general integral represents the aggregate of the envelopes of every one-parameter family of surfaces obtainable from the two-parameter family given by a complete integral.

With ϕ an arbitrary function, it is, in general, not possible actually to eliminate α from equations (8). Thus we usually cannot get a single equation, involving an arbitrary function, that represents the general integral of a non-linear equation of first order, as we could for Lagrange equations. Moreover, a complete integral is here not a particular case of the general integral, as is the case for linear equations.

For each particular choice of ϕ and α, equations (8) yield two particular surfaces, and therefore equations (8) considered simultaneously represent a space curve. Such a curve is called a *characteristic curve* (cf. Art. 38). Suppose a particular ϕ chosen and α regarded as a parameter, so that the eliminant obtainable from (8), a special case of the general integral, is the envelope of the corresponding family (7). Then this envelope is tangent to each member of the family (7) along a characteristic curve. The general integral may consequently be regarded as the aggregate of surfaces generated by characteristic curves.

The question of the possible existence of special integrals, analogous to the special integrals of the Lagrange linear equation (Art. 39), will not be discussed here, as it is more a matter of theoretical interest than of practical importance in a first course.

These definitions and geometric interpretations will be illustrated in connection with various specific non-linear equations in our later work, and the student should refer back to this article until the concepts are clearly understood.

45. Charpit's method. The basic idea in Charpit's method of solving a given non-linear partial differential equation of first order

$$F(x, y, z, p, q) = 0, \tag{1}$$

is the determination of a second equation of the same type,

$$G(x, y, z, p, q) = 0, \tag{2}$$

such that equations (1) and (2) can be solved for p and q in terms of x, y, and z, and such that these resulting expressions, inserted in

$$p(x, y, z)\, dx + q(x, y, z)\, dy - dz = 0, \tag{3}$$

make (3) integrable.

In order that (1) and (2) be solvable for p and q, these relations must be independent, and therefore their Jacobian (Art. 24)

$$J \equiv \frac{\partial(F, G)}{\partial(p, q)} \equiv F_pG_q - F_qG_p \tag{4}$$

cannot vanish identically. We shall suppose, in what follows, that $J \not\equiv 0$.

The necessary and sufficient condition that the total differential equation

$$P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz = 0$$

be integrable is that

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

vanish identically (Art. 19). Hence, if (3) is to be integrable, we must have, since here $P = p$, $Q = q$, $R = -1$,

$$p\frac{\partial q}{\partial z} - q\frac{\partial p}{\partial z} - \frac{\partial p}{\partial y} + \frac{\partial q}{\partial x} = 0. \tag{5}$$

We shall use this condition (5) to determine the required relation (2).

Regarding p and q in (1) as functions of x, y, z obtainable from (1) and (2), we get by differentiating (1) partially with respect to x (holding y and z fixed),

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0. \tag{6}$$

Similarly, partial differentiation of (2) with respect to x gives us

$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial G}{\partial q}\frac{\partial q}{\partial x} = 0. \tag{7}$$

If we solve (6) and (7) for $\partial q/\partial x$, for insertion in (5), we find

$$\frac{\partial q}{\partial x} = \frac{F_xG_p - F_pG_x}{J}. \tag{8}$$

Evidently $\partial q/\partial x$ can be so obtained, since by supposition $J \not\equiv 0$.

In the same way, we differentiate (1) and (2) partially with respect to y, getting

$$F_y + F_p\frac{\partial p}{\partial y} + F_q\frac{\partial q}{\partial y} = 0,$$

$$G_y + G_p\frac{\partial p}{\partial y} + G_q\frac{\partial q}{\partial y} = 0,$$

and, solving these equations for $\partial p/\partial y$, there is found

$$\frac{\partial p}{\partial y} = \frac{F_qG_y - F_yG_q}{J}. \tag{9}$$

Finally, differentiate (1) and (2) partially with respect to z, whence

$$F_z + F_p \frac{\partial p}{\partial z} + F_q \frac{\partial q}{\partial z} = 0,$$

$$G_z + G_p \frac{\partial p}{\partial z} + G_q \frac{\partial q}{\partial z} = 0,$$

and solve these equations for $\partial p/\partial z$ and $\partial q/\partial z$. There is obtained

$$\frac{\partial p}{\partial z} = \frac{F_q G_z - F_z G_q}{J}, \qquad \frac{\partial q}{\partial z} = \frac{F_z G_p - F_p G_z}{J}. \tag{10}$$

By inserting the expressions for $\partial q/\partial x$, $\partial p/\partial y$, $\partial p/\partial z$ and $\partial q/\partial z$, as given by (8)–(10), in (5), and by multiplying throughout by J, we get

$$p(F_z G_p - F_p G_z) - q(F_q G_z - F_z G_q) - (F_q G_y - F_y G_q) + (F_x G_p - F_p G_x) = 0,$$

or, upon rearrangement,

$$(F_x + pF_z) \frac{\partial G}{\partial p} + (F_y + qF_z) \frac{\partial G}{\partial q}$$
$$- (pF_p + qF_q) \frac{\partial G}{\partial z} - F_p \frac{\partial G}{\partial x} - F_q \frac{\partial G}{\partial y} = 0. \tag{11}$$

This is a six-dimensional homogeneous linear partial differential equation for the determination of G as a function of x, y, z, p, and q (Art. 41). Its subsidiary equations are

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}, \; dG = 0. \tag{12}$$

Our problem of solving the given equation (1) may therefore be attacked as follows. We first form the subsidiary equations (12). Since any integral of these equations is a solution of (11), we try to find one integral of (12), the simpler the better, containing p or q or both, say

$$u(x, y, z, p, q) = \alpha, \tag{13}$$

where α is an arbitrary constant. This gives us the desired relation (2), namely, $G \equiv u - \alpha = 0$. We then solve (1) and (13) simultaneously for p and q as functions of x, y, z, and α, substitute in (3), and integrate to get

$$f(x, y, z, \alpha, \beta) = 0, \tag{14}$$

where β is a second arbitrary constant.

Evidently the solution (14) so found, containing two arbitrary constants, will serve as a complete integral of equation (1). A singular integral, if one exists, and particular cases of the general integral may then be found as indicated in Art. 44.

It should be particularly remembered that, in the formation of the subsidiary equations (12), F_x denotes the partial derivative of F with respect to x when holding z as well as y fixed, etc.

Example 1. Find a complete integral and a singular integral, if one exists, of the non-linear equation

$$pq + x(2y + 1)p + (y^2 + y)q - (2y + 1)z = 0. \tag{15}$$

Solution. Here we have

$$F_x = (2y + 1)p, \quad F_y = 2xp + (2y + 1)q - 2z, \quad F_z = -(2y + 1),$$

$$F_p = q + x(2y + 1), \quad F_q = p + y^2 + y.$$

Hence the first denominator in the subsidiary equations (12) is $F_x + pF_z = 0$, and therefore we have $dp = 0$,

$$p = \alpha. \tag{16}$$

This will serve as the needed integral. Substituting α for p in the given partial differential equation (15), and solving for q, we get

$$q = \frac{(2y + 1)(z - \alpha x)}{y^2 + y + \alpha}. \tag{17}$$

Then (3) gives us

$$\alpha\, dx + \frac{(2y + 1)(z - \alpha x)}{y^2 + y + \alpha}\, dy - dz = 0,$$

or

$$\frac{dz - \alpha\, dx}{z - \alpha x} = \frac{(2y + 1)\, dy}{y^2 + y + \alpha},$$

which is evidently integrable. Consequently

$$\log(z - \alpha x) = \log(y^2 + y + \alpha) + \log \beta,$$

and

$$z = \alpha x + \beta(y^2 + y + \alpha). \tag{18}$$

This is the required complete integral. If we differentiate (18) partially with respect to α and β in turn, we find

$$0 = x + \beta, \quad 0 = y^2 + y + \alpha. \tag{19}$$

Elimination of α and β from (18) and (19) yields the equation

$$z = -x(y^2 + y). \tag{20}$$

It is easily verified that (20) satisfies (15), and consequently (20) is a singular integral. Evidently this third-degree equation (20) represents a surface distinct from any of the quadric surfaces in the two-parameter family (18), so that the singular integral is not a part of the complete integral.

In some problems it will be found advantageous to change the subsidiary equations (14) into a more convenient form by making use of the given differential equation, as in the following example.

Example 2. Find a complete integral of the equation

$$2zq^2 - y^2p + y^2q = 0. \tag{21}$$

Solution. In this case we have

$$F_x = 0, \quad F_y = 2y(q - p), \quad F_z = 2q^2, \quad F_p = -y^2, \quad F_q = 4zq + y^2,$$

and the subsidiary equations are

$$\frac{dp}{2pq^2} = \frac{dq}{2y(q - p) + 2q^3} = \frac{dz}{y^2p - 4zq^2 - y^2q} = \frac{dx}{y^2} = \frac{dy}{-4zq - y^2}. \tag{22}$$

To obtain an integral from these equations as they stand is not as easy as it was in Example 1. But we note that the denominator of the third fraction in (22) may, by virtue of the original equation (21), be replaced by $-2zq^2$. Making this substitution, the first and third ratios in (22) now yield the relation

$$\frac{dp}{2pq^2} = \frac{dz}{-2zq^2},$$

whence

$$\frac{dp}{p} + \frac{dz}{z} = 0,$$

$$\log p + \log z = \log \alpha,$$

$$p = \frac{\alpha}{z}. \tag{23}$$

Replacing p in (21) by α/z, we then get

$$2z^2q^2 + y^2zq - \alpha y^2 = 0,$$

from which there is found

$$q = \frac{-y^2 \pm y\sqrt{y^2 + 8\alpha}}{4z}. \tag{24}$$

Substituting from (23) and (24) into (3), we obtain

$$4\alpha\, dx - y^2\, dy \pm y\sqrt{y^2 + 8\alpha}\, dy - 4z\, dz = 0,$$

and integration yields

$$4\alpha x - \frac{y^3}{3} \pm \frac{1}{3}(y^2 + 8\alpha)^{3/2} - 2z^2 = \beta.$$

Rationalization then gives us the complete integral

$$(6z^2 - 12\alpha x + y^3 + 3\beta)^2 = (y^2 + 8\alpha)^3. \tag{25}$$

It is readily found that there exists no singular integral.

EXERCISES

Find a complete integral of each of the following non-linear equations. Find also any singular integrals that may exist.

1. $pq + xp + yq - z = 0.$
2. $p^2 - q^2 + 4xp + 4yq - 4z = 0.$
3. $pq + p + q = 0.$
4. $pq - p - q - 1 = 0.$
5. $q^2 - 2q + 3p = 1.$
6. $p^2 + 2pq + q^2 + xp + yq = z.$
7. $x^2p^2 + y^2q^2 = 1.$
8. $xp^2 + yq^2 = 1.$
9. $p^2 + q^2 = z^2.$
10. $pq + z = 1.$
11. $2p^2 - 2q^2 - 3x + 3y = 0.$
12. $z^2(p^2 - q^2) = 1.$
13. $pq + xp + yq = 0.$
14. $p^2 - q^2 + (x + y)^2 = 0.$
15. $p^2 + pq + x - y = 0.$
16. $p^2 + 4x^2q = 0.$
17. $pq = 4xy.$
18. $2z^2pq + x = 0.$
19. $y^4q^2 + x^2p = 0.$
20. $zq^2 = 18y^2p.$

46. Equations of the form $F(p, q) = 0$. A second relation connecting x, y, z, p, and q can always be found by Charpit's method for non-linear equations of certain types. In this and the following three articles we shall examine some equations of special form for which standard methods, obtainable from the general Charpit procedure, may be formulated. We call the four types of equations to be considered, *standard forms*.

Consider first a non-linear equation of the form

$$F(p, q) = 0, \tag{1}$$

containing p and q but none of the variables x, y, z. Since $F_x = 0$, $F_y = 0$, and $F_z = 0$, the denominators of the first two ratios in the Charpit subsidiary equations,

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}, \tag{2}$$

vanish identically, and therefore we have $dp = 0$ and $dq = 0$. We may thus use either $p = \alpha$, together with the value of q obtained from (1) when p is replaced by α, or $q = \alpha$, together with the result of solving $F(p, \alpha) = 0$ for p, whichever is more convenient.

Example. Find a complete integral of the differential equation

$$p^3 - 2pq + 3q = 5.$$

Solution. Since the equation is linear in q but of the third degree in p, it is simpler to use $p = \alpha$, whence

$$q = \frac{5 - \alpha^3}{3 - 2\alpha}.$$

Therefore we get, from $dz = p\,dx + q\,dy$,

$$dz = \alpha\,dx + \frac{5 - \alpha^3}{3 - 2\alpha}\,dy,$$

$$z = \alpha x + \frac{5 - \alpha^3}{3 - 2\alpha}\,y + \beta.$$

There is no singular integral.

Evidently every equation of type (1) will have as a complete integral a two-parameter family of planes, as in the above example. Moreover, the general integral of an equation of type (1) will be made up of developable surfaces (cf. Art. 29, Example 3). For, as stated in Art. 44, the general integral represents the aggregate of the envelopes of every one-parameter family of surfaces obtainable from a complete integral, and the envelope of a one-parameter family of planes is a developable surface (Art. 26).

47. Equations of the form $F(z, p, q) = 0$. Consider next a nonlinear equation of the form

$$F(z, p, q) = 0, \tag{1}$$

in which the independent variables x and y do not appear. Since $F_x = 0$, $F_y = 0$, the Charpit subsidiary equations now reduce to

$$\frac{dp}{pF_z} = \frac{dq}{qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}, \tag{2}$$

and we therefore always get, from the first two ratios,

$$q = \alpha p. \tag{3}$$

Combining (1) and (3), we then may find p and q in terms of z, say

$$p = f(z), \qquad q = \alpha p = \alpha f(z). \tag{4}$$

Therefore $dz = p\,dx + q\,dy$ yields

$$\frac{dz}{f(z)} = dx + \alpha\,dy,$$

$$\int \frac{dz}{f(z)} + \beta = x + \alpha y. \tag{5}$$

It thus appears that, in a complete integral (5), z will be a function of the combination $x + \alpha y$. We may use this fact to formulate a method of solving an equation of type (1), as follows. Set

$$z = g(x + \alpha y) = g(u), \tag{6}$$

say, so that

$$p = \frac{\partial g}{\partial x} = \frac{dg}{du}, \qquad q = \frac{\partial g}{\partial y} = \alpha \frac{dg}{du}. \tag{7}$$

Then the partial differential equation (1) is transformed into the ordinary differential equation

$$F\left(z, \frac{dz}{du}, \alpha \frac{dz}{du}\right) = 0, \tag{8}$$

of the first order. The general solution of (8), usually obtainable by one of the methods of Chapter I, contains the arbitrary element α in (8) and a second arbitrary constant of integration β, and thus yields a complete integral of (1).

Example. Find a complete integral of the equation

$$z^2(p^2 + q^2 + 1) = 1.$$

Solution. Making the substitution $u = x + \alpha y$, together with the derived relations (7), we get

$$(1 + \alpha^2)\left(\frac{dz}{du}\right)^2 + 1 = \frac{1}{z^2},$$

$$(1 + \alpha^2)\left(\frac{dz}{du}\right)^2 = \frac{1 - z^2}{z^2},$$

$$\sqrt{1 + \alpha^2}\,\frac{z\,dz}{\sqrt{1 - z^2}} = \pm\,du,$$

$$-\sqrt{1 + \alpha^2}\,\sqrt{1 - z^2} = \pm\,(u + \beta),$$

and

$$(1 + \alpha^2)(1 - z^2) = (x + \alpha y + \beta)^2.$$

This is the desired complete integral. It is left to the student to show that the given differential equation also possesses the singular integrals $z = \pm 1$.

The substitution $u = x + \alpha y$ has the geometric interpretation of rotating the coordinate axes in the xy-plane through the angle arctan α and stretching the x- and y-coordinates in the ratio $\sqrt{1 + \alpha^2} : 1$. Since equation (6) represents a cylinder, the surfaces given by a complete integral of an equation of type (1) will therefore be cylinders with elements at an angle arctan α with the x-axis and parallel to the xy-plane. Thus, the complete integral found in the above example represents circular cylinders of unit radius and with their axes in the xy-plane; evidently such cylinders do have the planes $z = \pm 1$ as envelopes.

EXERCISES

Find a complete integral of each of the non-linear equations in Exercises 1–6. Discard any solution that does not have a real geometric interpretation.

1. $q^2 - 3q + p = 2.$

2. $p^2 + q^2 = 4.$

3. $q^2 - p^2 = 9.$

4. $q + \sin p = 0.$

5. $q^6 + 2p^2q^2 + 3p^3 = 0.$

6. $2q^4 + 3pq^2 + p^2 = 0.$

7–12. Taking $\beta = 0$ in the answers to Exercises 1–6, find the developable surface that serves as envelope to each resulting one-parameter family of planes, and verify that each envelope is a solution of the corresponding differential equation.

Find a complete integral of each of the non-linear equations in Exercises 13–20. Find also any singular integrals that may exist.

13. $pq = z.$

14. $pq = z^2.$

15. $p^2 + q^2 = z^4.$

16. $z^2pq = 1 + z^2.$

17. $z^2q^2 = p.$

18. $p^2 - 2zq - z^2 = 0.$

19. $z^2pq + zp + zq = 1.$

20. $p^2 + q^2 + z^2 = z^4.$

48. Equations of the form $f(x, p) = g(y, q)$. Suppose now that a given non-linear partial differential equation is expressible in the form

$$f(x, p) = g(y, q), \tag{1}$$

in which only x and p occur in one group of terms and only y and q appear in the remaining terms. Letting F denote $f(x, p) - g(y, q)$, we then have

$$F_x = f_x, \quad F_y = -g_y, \quad F_z = 0, \quad F_p = f_p, \quad F_q = -g_q,$$

so that the Charpit subsidiary equations become

$$\frac{dp}{f_x} = \frac{dq}{-g_y} = \frac{dz}{-pf_p + qg_q} = \frac{dx}{-f_p} = \frac{dy}{g_q}. \tag{2}$$

From the first and fourth of these ratios we have

$$f_x\,dx + f_p\,dp = 0, \tag{3}$$

and from the second and fifth,

$$g_y\,dy + g_q\,dq = 0. \tag{4}$$

But the left members of (3) and (4) are respectively the total differentials df and dg of the functions $f(x, p)$ and $g(y, q)$ (Art. 19). Hence $f(x, p) = $ const. and $g(y, q) = $ const., and, by the original equation (1), these constants are equal. Therefore

$$f(x, p) = \alpha, \qquad g(y, q) = \alpha. \tag{5}$$

The procedure of solving an equation of type (1) is thus clear. We set each member of (1) equal to an arbitrary constant α, and solve the resulting pair of equations (5) for p in terms of x and α and for q in terms of y and α. We then insert these expressions for p and q in $p\,dx + q\,dy - dz = 0$, to get a total differential equation

$$dz = p(x, \alpha)\,dx + q(y, \alpha)\,dy. \tag{6}$$

This is evidently integrable, whence we find

$$z = \int p(x, \alpha)\,dx + \int q(y, \alpha)\,dy + \beta, \tag{7}$$

a complete integral of the original equation (1).

Example. Find a complete integral of the non-linear equation

$$(1 - x^2)yp^2 + x^2q = 0.$$

Solution. It is easy to see that this equation can be put into the form (1); we get

$$\frac{(1 - x^2)p^2}{x^2} = -\frac{q}{y} = \alpha.$$

Consequently

$$p = \pm\frac{\sqrt{\alpha}\,x}{\sqrt{1 - x^2}}, \qquad q = -\alpha y,$$

$$dz = \pm\frac{\sqrt{\alpha}\,x\,dx}{\sqrt{1 - x^2}} - \alpha y\,dy,$$

$$z = \mp\sqrt{\alpha(1 - x^2)} - \frac{\alpha y^2}{2} + \beta,$$

and

$$(2z + \alpha y^2 - 2\beta)^2 = 4\alpha(1 - x^2).$$

This is the required complete integral. There is no singular integral.

49. Equations of the form $z = xp + yq + f(p, q)$. Clairaut's ordinary differential equation (Art. 8) has as its analogue, in the field of partial differential equations, the type

$$z = xp + yq + f(p, q). \tag{1}$$

If we let $F \equiv xp + yq + f(p, q) - z = 0$, we have

$$F_x = p, \quad F_y = q, \quad F_z = -1, \quad F_p = x + f_p, \quad F_q = y + f_q,$$

whence $F_x + pF_z = 0$ and $F_y + qF_z = 0$. Consequently the Charpit subsidiary equations yield $dp = 0$ and $dq = 0$, so that we may take either $p =$ const. or $q =$ const. in conjunction with the given equation (1).

However, it is easier to make use of both integrals of the subsidiary equations. If we replace p and q in (1) by arbitrary constants α and β, respectively, we get

$$z = \alpha x + \beta y + f(\alpha, \beta). \tag{2}$$

Since this relation correctly yields $p = \alpha$ and $q = \beta$ upon partial differentiation, (2) is a solution of (1); and since (2) contains two arbitrary constants, it is a complete integral of (1).

Usually a non-linear equation of type (1) possesses a singular integral, the envelope of the two-parameter family of planes represented by the complete integral (2). As in the case of partial differential equations of the form $F(p, q) = 0$ (Art. 46), any solution belonging to the general integral of (1), being derived as the envelope of a one-parameter family of planes obtained from (2), will be a developable surface.

Example. Find a complete integral and the singular integral of the non-linear equation

$$z = xp + yq + p^2 - q^2. \tag{3}$$

Find also a developable surface belonging to the general integral of this partial differential equation.

Solution. Replacing p by α and q by β in the given equation (3), we have immediately a complete integral,

$$z = \alpha x + \beta y + \alpha^2 - \beta^2. \tag{4}$$

Partial differentiation, with respect to α and β in turn, of this equation, yields $0 = x + 2\alpha$, $0 = y - 2\beta$, whence $\alpha = -x/2$, $\beta = y/2$. Eliminating α and β from (4), we then get

$$4z = y^2 - x^2. \tag{5}$$

This is easily verified as a solution of (3), and is therefore a singular integral; the planes of the complete integral (4) have the hyperbolic paraboloid (5) as envelope.

To get a developable surface satisfying (3), we may set β in (4) equal to any desired function of α and then seek an envelope of the resulting one-parameter family of planes. For simplicity, set $\beta = 0$, so that (4) becomes

$$z = \alpha x + \alpha^2, \tag{6}$$

a family of planes parallel to the y-axis. Differentiating (6) partially with respect to α, we get $0 = x + 2\alpha$, or $\alpha = -x/2$, and elimination of α from (6) yields

$$4z + x^2 = 0. \tag{7}$$

This parabolic cylinder, with its elements parallel to the y-axis, is evidently a developable surface, and the envelope of the planes (6).

50. Equations reducible to standard forms. It naturally happens that many non-linear partial differential equations of the first order do not fall under any of the four standard forms discussed in Arts. 46–49, nor are their Charpit subsidiary equations readily solved.

Sometimes, however, it is possible to make a change of variable which is such that the transformed partial differential equation can be easily attacked. Obviously, no general rules can be given for choosing a suitable transformation in each case; we mention here only a few types of equations, reducible in each case to one of the standard forms.

(*a*) An equation of the form

$$F_1(x^m p, y^n q) = 0 \tag{1}$$

where m and n are any constants, can always be transformed into an equation of the type considered in Art. 46.* For, when $m \neq 1$, we may set $x_1 = x^{1-m}$, so that

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial x_1}\frac{dx_1}{dx} = (1-m)x^{-m}\frac{\partial z}{\partial x_1},$$

or

$$x^m p = (1-m)\frac{\partial z}{\partial x_1} \equiv (1-m)p_1; \tag{2}$$

* Theoretically, it is always possible to solve an equation of the form (1) for one of its arguments in terms of the other, thereby producing an equation of the standard form $f(x, p) = g(y, q)$ (Art. 48). However, it is sometimes not convenient or desirable to do this. Moreover, the transformations discussed in the present case (*a*) have considerable utility in connection with cases (*c*) and (*d*) considered later. Analogous remarks apply to equations of the form (6), under case (*b*).

and if $m = 1$, we let $x_1 = \log x$, whence we get

$$xp = \frac{\partial z}{\partial x_1} \equiv p_1. \tag{3}$$

Similarly, for $n \neq 1$, let $y_1 = y^{1-n}$, from which we find

$$y^n q = (1 - n)\frac{\partial z}{\partial y_1} \equiv (1 - n)q_1; \tag{4}$$

and when $n = 1$, let $y_1 = \log y$, so that

$$yq = q_1. \tag{5}$$

The use of the proper substitutions evidently converts equation (1) into a relation involving only p_1 and q_1, for which a complete integral may be found as indicated in Art. 46. Replacing x_1 and y_1 in this integral by their expressions in terms of x and y then yields a complete integral of partial differential equation (1).

(b) An equation of the form

$$F_1(z^k p, z^k q) = 0, \tag{6}$$

where k is any constant, may also be transformed into the type discussed in Art. 46. If $k \neq -1$, set $z_1 = z^{k+1}$, so that

$$p_1 \equiv \frac{\partial z_1}{\partial x} = \frac{\partial z}{\partial x}\frac{dz_1}{dz} = (k+1)z^k p, \tag{7}$$

$$q_1 \equiv \frac{\partial z_1}{\partial y} = \frac{\partial z}{\partial y}\frac{dz_1}{dz} = (k+1)z^k q. \tag{8}$$

Substituting from (7) and (8) into (6), we clearly get the stated standard form in p_1 and q_1. If $k = -1$, set $z_1 = \log z$, whence

$$p_1 \equiv \frac{\partial z_1}{\partial x} = \frac{p}{z}, \qquad q_1 \equiv \frac{\partial z_1}{\partial y} = \frac{q}{z},$$

and again (6) is reducible to the form $F(p_1, q_1) = 0$.

(c) An equation of the form

$$F_1(x^m z^k p, y^n z^k q) = 0 \tag{9}$$

may be transformed into the standard form $F(p_1, q_1) = 0$ by combining suitably chosen transformations from cases (a) and (b).

(d) An equation of the form

$$F_1(z, x^m p, y^n q) = 0 \tag{10}$$

may be transformed into the standard form $F(z, p_1, q_1) = 0$ of Art. 47 by means of the proper substitutions of case (a).

(e) An equation of the form

$$f_1(x, z^k p) = g_1(y, z^k q) \tag{11}$$

may be transformed into the standard form $f(x, p_1) = g(y, q_1)$ of Art. 48 by means of the proper substitution of case (b).

Example. Find a complete integral of the equation

$$2x^4p^2 - yzq - 3z^2 = 0.$$

Solution. As it stands, this equation does not fall under any of our standard forms, nor do the Charpit subsidiary equations,

$$\frac{dp}{8x^3p^2 - ypq - 6zp} = \frac{dq}{-zq - yq^2 - 6zq} = \frac{dz}{-4x^4p^2 + yzq} = \frac{dx}{-4x^4p} = \frac{dy}{yz},$$

offer any simple method of attack. But we may write the given equation as

$$2\left(\frac{x^2p}{z}\right)^2 - \frac{yq}{z} - 3 = 0,$$

from which it is apparent that it is an instance of case (c), with $k = -1$, $m = 2$, $n = 1$. Accordingly, we make the transformation

$$x_1 = x^{-1}, \quad y_1 = \log y, \quad z_1 = \log z.$$

Then

$$p_1 = \frac{\partial z_1}{\partial x_1} = \frac{\dfrac{dz_1}{dz}\dfrac{\partial z}{\partial x}}{\dfrac{dx_1}{dx}} = \frac{\dfrac{1}{z}p}{-x^{-2}} = -\frac{x^2p}{z},$$

$$q_1 = \frac{\partial z_1}{\partial y_1} = \frac{\dfrac{dz_1}{dz}\dfrac{\partial z}{\partial y}}{\dfrac{dy_1}{dy}} = \frac{\dfrac{1}{z}q}{\dfrac{1}{y}} = \frac{yq}{z},$$

and our equation becomes

$$2p_1^2 - q_1 - 3 = 0.$$

This is of the form $F(p_1, q_1) = 0$. Taking $p_1 = \alpha$, we get $q_1 = 2\alpha^2 - 3$,

$$dz_1 = \alpha\, dx_1 + (2\alpha^2 - 3)\, dy_1,$$

$$z_1 = \alpha x_1 + (2\alpha^2 - 3)y_1 + \beta.$$

Replacing x_1 by $1/x$, y_1 by $\log y$, and z_1 by $\log z$, we therefore have the desired complete integral,

$$x \log z = \alpha + (2\alpha^2 - 3)x \log y + \beta x.$$

EXERCISES

In Exercises 1–6, find a complete integral of each of the given partial differential equations.

1. $xp - y^2q^2 = 1.$

2. $yp - x^2q^2 = x^2y.$

3. $pq + p = x^2y^2.$

4. $p^2q - 2x^2q = x^2y.$

5. $ypq + xyq - 4yp = x(1 + 4y).$

6. $pq^2 - 2pq - 3q^2 - yp + 6q = xy.$

In Exercises 7–12, find the singular integral of each of the given equations. Also find, in each case, a developable surface belonging to the general integral.

7. $z = xp + yq + 3pq.$

8. $z = xp + yq - 2p^2 - 3q^2.$

9. $z = xp + yq - 4p^2q^2.$

10. $xp^2 + ypq - zp + 1 = 0.$

11. $xp^2q + ypq^2 - zpq + 1 = 0.$

12. $(xp + yq - z)(p^2 + q^2) = pq.$

In Exercises 13–18, find a complete integral of each of the given equations, using the transformations of Art. 50.

13. $x^2yp^2 - xpq + q = 0.$

14. $z^4q^2 - z^2p = 1.$

15. $x^2p^2 + xpq - z^2 = 0.$

16. $xz^2q^2 - 2yz^2pq + 2xy^2 = 0.$

17. $p^2 + x^2y^2q^2 = x^2z^2.$

18. $z^2p^2 - x^4y^2zq = x^4.$

19. Using the transformation $x_1 = e^x$, $y_1 = e^y$, $z_1 = e^z$, find a complete integral of the equation $e^{2y}p^2 - e^{2x}q^2 = e^{2(x+y-z)}$.

20. Using a suitable transformation, find a complete integral of the equation $pq + z^2 \cos x \sin y = 0.$

51. Jacobi's method. When a non-linear partial differential equation of first order involves more than two independent variables, either explicitly or implicitly through the presence of partial derivatives, a method other than Charpit's must be used to solve the equation. We consider, in this article, Jacobi's method of attacking such problems.

Let $x_1, x_2, \cdots, x_n$ be the independent variables, z the dependent variable, and $p_j \equiv \partial z/\partial x_j$ $(j = 1, 2, \cdots, n)$ the partial derivatives concerned, in the given equation,

$$G(x_1, x_2, \cdots, x_n, z, p_1, p_2, \cdots, p_n) = 0. \tag{1}$$

We need consider only those partial differential equations into which the dependent variable does not enter explicitly. For, if

$$u \equiv g(x_1, x_2, \cdots, x_n, z) = 0 \tag{2}$$

is a solution of (1), we have (Art. 20)

$$p_j \equiv \frac{\partial z}{\partial x_j} = -\frac{\dfrac{\partial u}{\partial x_j}}{\dfrac{\partial u}{\partial z}}, \qquad (j = 1, 2, \cdots, n)$$

and if we set $z = x_{n+1}$, $P_j = \partial u/\partial x_j$ $(j = 1, 2, \cdots, n + 1)$, we get

$$p_j = -\frac{P_j}{P_{n+1}}, \qquad (j = 1, 2, \cdots, n) \tag{3}$$

whence (1) takes the form

$$F(x_1, x_2, \cdots, x_n, x_{n+1}, P_1, P_2, \cdots, P_n, P_{n+1}) = 0. \tag{4}$$

This is a partial differential equation in $n + 1$ variables, in which the dependent variable u does not appear. Thus, any equation of the form (1) may be transformed into another equation of type (4), lacking the dependent variable but containing one more independent variable (cf. Art. 41). If we can find a solution $u = 0$ of (4), replacement of x_{n+1} by z yields the solution (2) of the original equation (1).

For example, a partial differential equation with two independent variables x and y, of the form treated earlier in this chapter,

$$F(x, y, z, p, q) = 0,$$

can be transformed into an equation in three independent variables by setting $x = x_1$, $y = x_2$, $z = x_3$, $p = -p_1/p_3$, and $q = -p_2/p_3$, where $p_j \equiv \partial u/\partial x_j$ $(j = 1, 2, 3)$. This gives us an equation, say

$$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = 0,$$

in which the new dependent variable u does not appear.

The method now under consideration thus applies not only to equations involving three or more independent variables and one dependent variable, but also to those equations for which Charpit's method may be used.

For simplicity of treatment, we now confine ourselves to a discussion of the equation

$$F(x_1, x_2, x_3, p_1, p_2, p_3) = 0, \tag{5}$$

containing only three independent variables, x_1, x_2, and x_3, and the three partial derivatives $p_j \equiv \partial z/\partial x_j$ $(j = 1, 2, 3)$.

Jacobi's method, applied to equation (5), consists of trying to find *two* additional relations of similar form,

$$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1, \tag{6}$$

$$F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2, \tag{7}$$

where a_1 and a_2 are arbitrary constants, such that p_1, p_2, and p_3 can be determined from (5)–(7) as functions of the x's and a's, and such that these functions, when inserted in the differential relation

$$dz = p_1\, dx_1 + p_2\, dx_2 + p_3\, dx_3, \tag{8}$$

yield an integrable equation. The result of integrating (8), whereby a third arbitrary constant a_3 is introduced, is called a *complete integral* of (5).

To evolve a method of determining the needed relations (6) and (7), having the requisite properties, we proceed as follows. Regarding p_1, p_2, and p_3 in (5) as the functions of x_1, x_2, x_3 obtained from (5)–(7), partial differentiation of (5) with respect to x_1, holding x_2 and x_3 fixed, gives us

$$\frac{\partial F}{\partial x_1} + \frac{\partial F}{\partial p_1}\frac{\partial p_1}{\partial x_1} + \frac{\partial F}{\partial p_2}\frac{\partial p_2}{\partial x_1} + \frac{\partial F}{\partial p_3}\frac{\partial p_3}{\partial x_1} = 0. \tag{9}$$

Likewise, partial differentiation of (6) with respect to x_1 yields

$$\frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial p_1}\frac{\partial p_1}{\partial x_1} + \frac{\partial F_1}{\partial p_2}\frac{\partial p_2}{\partial x_1} + \frac{\partial F_1}{\partial p_3}\frac{\partial p_3}{\partial x_1} = 0. \tag{10}$$

To eliminate $\partial p_1/\partial x_1$ from (9) and (10), multiply (9) by $\partial F_1/\partial p_1$ and (10) by $\partial F/\partial p_1$, and subtract; we get

$$\frac{\partial(F, F_1)}{\partial(x_1, p_1)} + \frac{\partial(F, F_1)}{\partial(p_2, p_1)}\frac{\partial p_2}{\partial x_1} + \frac{\partial(F, F_1)}{\partial(p_3, p_1)}\frac{\partial p_3}{\partial x_1} = 0, \tag{11}$$

where

$$\frac{\partial(F, F_1)}{\partial(x_1, p_1)} \equiv \frac{\partial F}{\partial x_1}\frac{\partial F_1}{\partial p_1} - \frac{\partial F}{\partial p_1}\frac{\partial F_1}{\partial x_1},$$

etc., are Jacobians (Art. 24).

Similarly, partial differentiation of (5) and (6) with respect to x_2, and elimination of $\partial p_2/\partial x_2$ from the resulting two relations, give us

$$\frac{\partial(F, F_1)}{\partial(x_2, p_2)} + \frac{\partial(F, F_1)}{\partial(p_1, p_2)}\frac{\partial p_1}{\partial x_2} + \frac{\partial(F, F_1)}{\partial(p_3, p_2)}\frac{\partial p_3}{\partial x_2} = 0; \tag{12}$$

partial differentiation of (5) and (6) with respect to x_3, and elimination of $\partial p_3/\partial x_3$, yield

$$\frac{\partial(F, F_1)}{\partial(x_3, p_3)} + \frac{\partial(F, F_1)}{\partial(p_1, p_3)}\frac{\partial p_1}{\partial x_3} + \frac{\partial(F, F_1)}{\partial(p_2, p_3)}\frac{\partial p_2}{\partial x_3} = 0. \tag{13}$$

Now, in order that (8) be integrable, that is, in order that $p_1\,dx_1 + p_2\,dx_2 + p_3\,dx_3$ be the exact differential of the z-function sought, it is necessary (cf. equations (4)–(6), Art. 19, with $\lambda = 1$), that

$$\frac{\partial p_1}{\partial x_2} = \frac{\partial^2 z}{\partial x_2 \partial x_1} = \frac{\partial^2 z}{\partial x_1 \partial x_2} = \frac{\partial p_2}{\partial x_1},$$

$$\frac{\partial p_2}{\partial x_3} = \frac{\partial p_3}{\partial x_2}, \qquad \frac{\partial p_3}{\partial x_1} = \frac{\partial p_1}{\partial x_3}. \tag{14}$$

Moreover, it is easy to verify that

$$\frac{\partial(F,\,F_1)}{\partial(p_2,\,p_1)} = -\frac{\partial(F,\,F_1)}{\partial(p_1,\,p_2)}, \qquad \frac{\partial(F,\,F_1)}{\partial(p_3,\,p_1)} = -\frac{\partial(F,\,F_1)}{\partial(p_1,\,p_3)},$$

$$\frac{\partial(F,\,F_1)}{\partial(p_3,\,p_2)} = -\frac{\partial(F,\,F_1)}{\partial(p_2,\,p_3)}. \tag{15}$$

Hence, if we add (11), (12), and (13), three pairs of terms cancel by virtue of relations (14) and (15), and the result is

$$\frac{\partial(F,\,F_1)}{\partial(x_1,\,p_1)} + \frac{\partial(F,\,F_1)}{\partial(x_2,\,p_2)} + \frac{\partial(F,\,F_1)}{\partial(x_3,\,p_3)} = 0, \tag{16}$$

or, in expanded form,

$$\frac{\partial F}{\partial x_1}\frac{\partial F_1}{\partial p_1} + \frac{\partial F}{\partial x_2}\frac{\partial F_1}{\partial p_2} + \frac{\partial F}{\partial x_3}\frac{\partial F_1}{\partial p_3} - \frac{\partial F}{\partial p_1}\frac{\partial F_1}{\partial x_1} - \frac{\partial F}{\partial p_2}\frac{\partial F_1}{\partial x_2} - \frac{\partial F}{\partial p_3}\frac{\partial F_1}{\partial x_3} = 0. \tag{17}$$

For simplicity, we write (16) in the compact symbolic form

$$(F, F_1) = 0. \tag{18}$$

If we treat (5) and (7) as we did (5) and (6), we are evidently led to

$$(F, F_2) = 0; \tag{19}$$

and, from (6) and (7), there is likewise found

$$(F_1, F_2) = 0. \tag{20}$$

The expanded form (17) is now revealed as a seven-dimensional homogeneous linear partial differential equation for the determination of F_1 (Art. 41). Its subsidiary equations are

$$\frac{dp_1}{\dfrac{\partial F}{\partial x_1}} = \frac{dp_2}{\dfrac{\partial F}{\partial x_2}} = \frac{dp_3}{\dfrac{\partial F}{\partial x_3}} = \frac{dx_1}{-\dfrac{\partial F}{\partial p_1}} = \frac{dx_2}{-\dfrac{\partial F}{\partial p_2}} = \frac{dx_3}{-\dfrac{\partial F}{\partial p_3}}. \tag{21}$$

These relations also serve as subsidiary equations to (19), for the determination of F_2.

If we obtain two independent integrals, $F_1 = a_1$ and $F_2 = a_2$, of the subsidiary equations (21), relations (18) and (19) will be fulfilled. In addition, our analysis shows that F_1 and F_2 must be such that equation (20) is satisfied. Accordingly, we may formulate the procedure as follows.

Given the partial differential equation (5), first form the subsidiary equations (21). Then find two independent integrals (6) and (7) of these subsidiary equations, and such that condition (20) is met. Solve (5)–(7) for p_1, p_2, and p_3, insert their expressions in (8), and integrate to get a complete integral of (5).

Example. Find a complete integral of the equation

$$x_2p_1p_2 - 3x_3^2p_1^2 + p_3 - 4 = 0.$$

Solution. The subsidiary equations (21) are here

$$dp_1 = 0, \quad \frac{dp_2}{p_1p_2} = \frac{dp_3}{-6x_3p_1^2} = \frac{dx_1}{-x_2p_2 + 6x_3^2p_1} = \frac{dx_2}{-x_2p_1} = \frac{dx_3}{-1}.$$

From the first equation, we get

$$F_1 \equiv p_1 = a_1,$$

and from the first and fourth ratios,

$$F_2 \equiv x_2p_2 = a_2.$$

It is readily found that $(F_1, F_2) = 0$, since each term vanishes. Substituting $p_1 = a_1$, $p_2 = a_2/x_2$ in the given equation, we also get $p_3 = 4 - a_1a_2 + 3a_1^2x_3^2$, and consequently (8) becomes

$$dz = a_1\,dx_1 + \frac{a_2}{x_2}\,dx_2 + (4 - a_1a_2 + 3a_1^2x_3^2)\,dx_3.$$

This is immediately integrable, and we get as the desired complete integral,

$$z = a_1x_1 + a_2\log x_2 + (4 - a_1a_2)x_3 + a_1^2x_3^3 + a_3.$$

The above process for solving an equation involving three independent variables can be extended to the case of n independent variables. The general theory is too long and complicated to be given here; we merely state the following rule.

Given the partial differential equation

$$F(x_1, x_2, \cdots, \quad x_n, p_1, p_2, \cdots, \quad p_n) = 0,$$

form the subsidiary equations

$$\frac{dp_1}{\frac{\partial F}{\partial x_1}} = \frac{dp_2}{\frac{\partial F}{\partial x_2}} = \cdots = \frac{dp_n}{\frac{\partial F}{\partial x_n}} = \frac{dx_1}{-\frac{\partial F}{\partial p_1}} = \frac{dx_2}{-\frac{\partial F}{\partial p_2}} = \cdots = \frac{dx_n}{-\frac{\partial F}{\partial p_n}}.$$

Try to find $n - 1$ independent integrals, $F_j = a_j$ $(j = 1, 2, \cdots, n - 1)$, of these subsidiary equations, such that the $\frac{1}{2}(n - 1)(n - 2)$ relations

$$\frac{\partial(F_j, F_k)}{\partial(x_1, p_1)} + \frac{\partial(F_j, F_k)}{\partial(x_2, p_2)} + \cdots + \frac{\partial(F_j, F_k)}{\partial(x_n, p_n)} = 0$$

$(j,\ k = 1,\ 2,\ \cdots,\ n - 1;\ j \neq k)$ are all satisfied. Then solve the n equations $F = 0$ and $F_j = a_j$ $(j = 1, 2, \cdots, n - 1)$ for the p's in terms of the x's and a's, and insert their expressions in

$$dz = p_1\,dx_1 + p_2\,dx_2 + \cdots + p_n\,dx_n.$$

Integration of this equation leads to a complete integral, involving n arbitrary constants.

EXERCISES

In Exercises 1–12, the references are to the exercises following Art. 45. Transform each of these equations to one involving three independent variables but no dependent variable, as described in Art. 51, and hence find a complete integral by Jacobi's method.

1. Ex. 1.	**2.** Ex. 4.	**3.** Ex. 5.	**4.** Ex. 6.
5. Ex. 7.	**6.** Ex. 10.	**7.** Ex. 12.	**8.** Ex. 13.
9. Ex. 14.	**10.** Ex. 15.	**11.** Ex. 16.	**12.** Ex. 17.

13. Show that the standard form $F(p, q) = 0$, Art. 46, can be transformed into an equation of the form $G(p_1, p_2, p_3) = 0$, and that such an equation is always solvable by Jacobi's method.

14. Show that the standard form $F(z, p, q) = 0$, Art. 47, can be transformed into an equation of the form $G(x_3, p_1, p_2, p_3) = 0$, and that this is always solvable by Jacobi's method.

15. Show that the standard form $f(x, p) = g(y, q)$, Art. 48, can be transformed into an equation of the form $\phi(x_1, p_1, p_3) = \psi(x_2, p_2, p_3)$, and that this is always solvable by Jacobi's method.

16. Show that the standard form $z = xp + yq + f(p, q)$, Art. 49, can be transformed into an equation of the form $x_1p_1 + x_2p_2 + x_3p_3 = g(p_1, p_2, p_3)$, and that this is always solvable by Jacobi's method.

Find a complete integral of each of the equations in Exercises 17–20.

17. $x_1p_1p_2 + p_3p_4 = 0$.

18. $x_1^2p_1^2 - x_2p_2 + x_3p_3 - x_4p_4 = 0$.

19. $p_3^2 + 2zp_1 + 2zp_2 = z^2$.

20. $x_3^2p_3^2 + x_1p_1 + x_2p_2 = z$.

52. Systems of equations. We consider next a system of two or more equations of the form discussed in the preceding article. Let the system be denoted by

$$\begin{aligned} F(x_1, x_2, \cdots, x_n, p_1, p_2, \cdots, p_n) &= 0, \\ F_1(x_1, x_2, \cdots, x_n, p_1, p_2, \cdots, p_n) &= 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad & \cdot \, , \\ F_m(x_1, x_2, \cdots, x_n, p_1, p_2, \cdots, p_n) &= 0. \end{aligned} \tag{1}$$

It should be particularly noticed that although the number n of independent variables may be three or more, the system involves only one dependent variable z. Thus, our problem is that of finding one functional relation between the x's and z, satisfying all $(m + 1)$ equations.

We first set aside the case in which the number $(m + 1)$ of equations exceeds n. When $m + 1 > n$, the p's may or may not be obtainable, in terms of the x's, from a certain n of the equations. If no set of n equations yields the p's, or if the p's can be so obtained but do not satisfy the remaining $m + 1 - n$ equations identically, the system (1) is said to be inconsistent. If the p's can be found from a certain set of n equations, and these values satisfy all the remaining equations, then the system (1) is consistent, at least in this respect, and any n of them will constitute an equivalent system.

Consequently we may suppose that $m < n$, and that the equations (1) are independent, that is, that the system cannot be reduced to an equivalent system containing fewer equations. For definiteness and simplicity we take $n = 3$, and consider various possibilities.

(*a*) Suppose that we have two equations,

$$\begin{aligned} F(x_1, x_2, x_3, p_1, p_2, p_3) &= 0, \\ F_1(x_1, x_2, x_3, p_1, p_2, p_3) &= 0. \end{aligned} \tag{2}$$

If $(F, F_1) = 0$, either identically or by virtue of one or both of the relations (2), we try to find a third equation, $F_2 = 0$, by Jacobi's method. To do this, we may use the equations subsidiary to either one of the given equations (2). As shown in Art. 51, the function F_2 should be such that $(F, F_2) = 0$ and $(F_1, F_2) = 0$. When we have found a suitable function F_2, we adjoin the equation $F_2 = 0$ to the given system, solve for p_1, p_2, and p_3, substitute in

$$dz = p_1\, dx_1 + p_2\, dx_2 + p_3\, dx_3, \tag{3}$$

and integrate. The resulting relation will then satisfy both of the original equations of our system.

(b) Suppose again that we are given the system (2) of two equations, but that (F, F_1) reduces to an expression F_2 involving at least one of the p's. Since (F, F_1) must be made to vanish in order that the total differential equation (3) be integrable (that is, in order that the given system have a solution), we set $F_2 = 0$ and use this in conjunction with (2) to determine the p's. Of course, the function F_2 so found must be such that $(F, F_2) = 0$ and $(F_1, F_2) = 0$; however, in practice it may be simpler to proceed directly to the determination of the p's and the integration of (3), when these steps are possible.

(c) Next suppose that, given the system of two equations (2), computation of (F, F_1) yields a non-zero expression free of p_1, p_2, and p_3. Whether this expression is a constant or a function of the x's, it cannot be made to vanish, and therefore the system (2) is inconsistent in the sense that it has no common solution.

(d) Suppose next that the given system consists of three independent equations,

$$
\begin{aligned}
F(x_1, x_2, x_3, p_1, p_2, p_3) &= 0, \\
F_1(x_1, x_2, x_3, p_1, p_2, p_3) &= 0, \qquad (4)\\
F_2(x_1, x_2, x_3, p_1, p_2, p_3) &= 0,
\end{aligned}
$$

and that (F, F_1), (F, F_2), and (F_1, F_2) all vanish either identically or by virtue of the relations (4). Then the system (4) may be solved for the p's directly, these values inserted in (3), and the latter equation integrated. As in (b), it is not necessary, in practice, to compute (F, F_1), (F, F_2), and (F_1, F_2); we may try directly to carry out the process of solving the system (4).

(e) Finally, suppose that the given system consists of three equations (4), but that at least one of the expressions (F, F_1), (F, F_2) (F_1, F_2) fails to vanish even with the aid of relations (4). Then our system is inconsistent, for no common solution of equations (4) is obtainable.

$dz = p_1\,dx_1 + p_2\,dx_2 + p_3\,dx_3.$

Example. Examine the system of equations

$$F \equiv x_1p_1p_2 - 2x_3p_2 + p_3 = 0, \qquad F_1 \equiv 2p_1 - p_3 + 3 = 0$$

for consistency, and find a common solution if one exists.

Solution. The first step in connection with a system of two equations is to compute (F, F_1), and thus determine whether the system falls under case (a), (b), or (c). Here we find

$$(F, F_1) = 2p_1p_2 + 2p_2 = 2p_2(p_1 + 1).$$

Hence our system is an instance of case (*b*), and can be made consistent by setting $(F, F_1) \equiv F_2 = 0$. Now $(F, F_1) \equiv F_2$ can be made to vanish either by taking $F_2 \equiv p_2 = 0$ or by taking $F_2 \equiv p_1 + 1 = 0$. With the former choice, $F = 0$ gives us $p_3 = 0$, whence $F_1 = 0$ yields $p_1 = -\frac{3}{2}$. Therefore

$$dz = -\tfrac{3}{2}\,dx_1, \qquad z = -\tfrac{3}{2}x_1 + a.$$

It is easily seen that $z = -\frac{3}{2}x_1 + a$ satisfies both $F = 0$ and $F_1 = 0$.

On the other hand, taking $F_2 \equiv p_1 + 1 = 0$, we get $p_3 = 1$ from $F_1 = 0$, and then $p_2 = 1/(x_1 + 2x_3)$ from $F = 0$. This leads to

$$dz = -\,dx_1 + \frac{dx_2}{x_1 + 2x_3} + dx_3,$$

which is not integrable.

The student should show that when $F_2 \equiv p_2 = 0$, both (F, F_2) and (F_1, F_2) vanish identically, whereas the choice $F_2 \equiv p_1 + 1 = 0$ leads to $(F, F_2) = p_1p_2 \neq 0$. This accounts for the integrability of the former ordinary differential equation and the non-integrability of the latter.

The analysis into five cases when $n = 3$ is typical, and serves as a pattern for the treatment of systems of equations involving more than three independent variables. Because of the greater complexity of the argument for $n > 3$, we shall not delve further into these questions. (See Exercises 8–10, below.)

EXERCISES

In Exercises 1–7, examine each of the given systems for consistency, and determine a common solution when possible.

1. $3x_2p_1^2 + 2p_2 + x_3p_3 = 1,\ p_1 - x_3p_3 = 1.$
2. $x_2p_1^2 - p_2 + 2x_2x_3^2p_3 = 0, \quad x_2p_1 + p_2 = 0.$
3. $x_1p_1 - x_2p_2 - (x_1 + x_2)x_3 = 0,\ p_1 - p_2 + 2x_1x_2p_3 - 2x_3 = 0.$
4. $p_3^2 + x_3^2(p_1 + p_2 - x_1 - x_2 - 4) = 0, \quad x_1p_1 + x_2p_2 = 2x_1x_2 + 1.$
5. $x_1p_1^2 - 2x_3p_2 + 4x_2p_3 = 0,\ 3x_1p_1^2 - p_2^2 - 2p_3^2 + x_2x_3^2 = 0.$
6. $x_1p_3^2 - p_1 - x_1x_2^2p_2 + 5x_1 = 0,\ 2p_1 - x_1x_2^2p_2 = 0,\ p_1 - 2x_1p_3 = 0.$
7. $x_3p_2p_3 - x_2p_1 - 3x_2x_3p_3 = 0,\ p_1^2 + x_3^2p_3 = 0,\ 2x_2p_1 - x_3p_2 = 0.$
8. Make an analysis, similar to that of Art. 52, for a system of two equations involving four independent variables. Hence classify under one of your cases and, if possible, find a common solution of the system

$$x_3p_1 - x_3p_2 - p_3 = 0, \qquad p_1^2 + x_4^2p_4 + 1 = 0.$$

9. Make an analysis for a system of three equations involving four independent variables. Hence classify under one of your cases and, if possible, find a common solution of the system

$$x_2p_1^2 - p_2 - x_2x_4^2p_4 = 0, \qquad x_3^3p_3 + 6x_4^2p_4 = 0,$$

$$3x_2p_1 + p_2 - x_2x_3^3p_3 + x_2 = 0.$$

10. Make an analysis for a system of four equations involving four independent variables. Hence classify under one of your cases and, if possible, find a common solution of the system

$$p_1 + 2x_1p_2 + x_3^2p_3 - x_4^2p_4 = 0, \qquad p_2 - x_4^2p_4 = 0,$$
$$p_1^2 + 4x_1^2x_3^2p_3 = 0, \qquad p_1^2 + 4x_1^2p_2 = 0.$$

53. Applications. We have already considered, at various points in this chapter, a few geometric aspects of non-linear partial differential equations of the first order and their solutions. In this article we shall illustrate, by means of examples, other geometric problems.

In a number of instances, the problem is to find a surface satisfying a given partial differential equation and possessing certain geometric properties. Various methods of solving the associated subsidiary equations may lead to different complete integrals, then particular cases of the general integral are obtainable from each complete integral as described in Art. 44, and, finally, the partial differential equation may possess a singular integral. The proper surface, meeting the stated geometric conditions, may belong to any one of the categories mentioned. However, the conditions of the problem often determine the correct type of solution, as in the following example.

Example 1. Find the surface satisfying the partial differential equation

$$z = xp + yq - 2p^2 - 3q^2, \tag{1}$$

and generated by characteristic curves parallel to the y-axis.

Solution. Since particular cases of the general integral and the singular integral (when one exists) are all derivable from a complete integral, the latter should be found first of all. Here the given partial differential equation (1) is of the standard form discussed in Art. 49, and consequently a complete integral is immediately found; it is

$$z = \alpha x + \beta y - 2\alpha^2 - 3\beta^2, \tag{2}$$

a two-parameter family of planes.

Now, in this problem, we are concerned with characteristic curves, which arise in connection with the general integral (Art. 44). Hence we should assume a functional relation between the arbitrary constants α and β, and determine the form of this relation from the geometric data. Setting $\beta = \phi(\alpha)$, (2) becomes

$$z = \alpha x + \phi y - 2\alpha^2 - 3\phi^2. \tag{3}$$

Differentiating this partially with respect to α, we get

$$0 = x + \phi' y - 4\alpha - 6\phi\phi'. \tag{4}$$

Equations (3) and (4) together define characteristic curves; since each equation is linear in x, y, and z, these characteristic curves are all straight lines.

Let a, b, and c, as functions of α, denote a set of direction numbers of the characteristic lines (3) – (4). Sets of direction numbers of the normals to the planes (3) and (4) are respectively $\alpha, \phi, -1$ and $1, \phi', 0$. Conditions for perpendicularity (Art. 22) then give us the relations

$$\alpha a + \phi b - c = 0, \qquad a + \phi' b = 0.$$

From these, we find as one set of direction numbers of the characteristic lines,

$$-\phi', \quad 1, \quad \phi - \alpha\phi'. \tag{5}$$

Now the y-axis has the direction numbers 0, 1, 0. In order that the characteristic lines be parallel to the y-axis, we must therefore have $-\phi' = 0, \phi - \alpha\phi' = 0$, whence $\phi(\alpha) = 0$.

Therefore a suitable one-parameter family of planes is obtained from (3) by setting $\phi = 0$:

$$z = \alpha x - 2\alpha^2. \tag{6}$$

The envelope of these planes is easily found to have the equation

$$8z = x^2. \tag{7}$$

This is a parabolic cylinder, with elements which also play the role of characteristic curves parallel to the y-axis. Accordingly, (7) represents the desired surface.

The two-parameter family of planes (2) has an envelope; the student should show that its equation is

$$24z = 3x^2 + 2y^2. \tag{8}$$

This elliptic paraboloid is the singular integral of the differential equation (1). The characteristic curves (3) – (4) are all tangent to the surface (8), and the parabolic cylinder (7) is tangent to the paraboloid (8) along the curve $8z = x^2$, $y = 0$ (Fig. 11).

It sometimes happens that different methods of integrating the Charpit or Jacobi subsidiary equations, or other considerations, lead to distinct complete integrals. For instance, the partial differential equation

$$z^2(p^2 + q^2 + 1) = 1, \tag{9}$$

discussed in the example of Art. 47, was found to have

$$(x + \alpha y + \beta)^2 + (1 + \alpha^2)(z^2 - 1) = 0, \tag{10}$$

a two-parameter family of circular cylinders of unit radius and with their axes in the xy-plane, as a complete integral. On the other hand, in Example 1 of Art. 29, the spheres

$$(x - a)^2 + (y - b)^2 + z^2 = 1, \tag{11}$$

of unit radius and with their centers in the xy-plane, were shown to have the differential equation (9) as their eliminant; accordingly, equation (11) may equally well be regarded as a complete integral of (9).

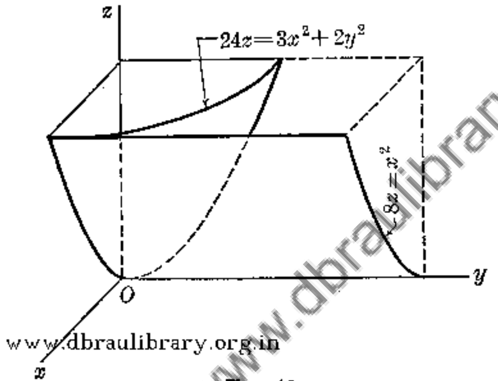

FIG. 11

To get a clue to the relationship between a pair of complete integrals, let us investigate the foregoing example still further. If we take $\beta = 0$ in (10), we get the one-parameter family of cylinders

$$(x + \alpha y)^2 + (1 + \alpha^2)(z^2 - 1) = 0. \tag{12}$$

By the usual method, it is found that the envelope of the cylinders (12), that is, a part of the general integral of (9), is

$$x^2 + y^2 + z^2 = 1,$$

which is a sphere obtainable from (11) by setting $a = b = 0$. Likewise, other choices of β as a function of α in the complete integral (10) will lead to particular cases of the general integral which are included in the complete integral (11).

Apparently, therefore, each complete integral may be regarded as a particular case of the general integral when the latter is derived from the other complete integral. Suppose that

$$f(x, y, z, \alpha, \beta) = 0 \tag{13}$$

and

$$g(x, y, z, a, b) = 0 \tag{14}$$

are two complete integrals of the same partial differential equation. Differentiate (13) and (14) with respect to x and y in turn, regarding z in each case as the function of x and y defined by the corresponding equation. This gives us four additional relations,

$$f_x = 0, \qquad f_y = 0, \qquad g_x = 0, \qquad g_y = 0. \tag{15}$$

If we can eliminate x, y, z, p, and q from the six equations (13) – (15), we get a functional relation between α and β and a and b, say

$$\beta = \phi(\alpha, a, b). \tag{16}$$

Now substitute this expression for β in (13) to get a one-parameter family, regarding α as parameter and a and b fixed. The particular case of the general integral thereby obtained should then be the complete integral (14).

This relationship between integrals gives justification to the title general integral. For, the aggregate of envelopes derived from any one complete integral will include, by virtue of equations such as (16), all other complete integrals (or particular cases of these complete integrals). Thus, this totality will include all solutions, with the possible exception of singular integrals, and therefore may rightly be called the general integral.

Example 2. Perform the elimination process described above, for the complete integrals (10) and (11) of the differential equation (9).

Solution. Taking

$$f \equiv (x + \alpha y + \beta)^2 + (1 + \alpha^2)(z^2 - 1) = 0, \tag{17}$$

$$g \equiv (x - a)^2 + (y - b)^2 + z^2 - 1 = 0, \tag{18}$$

we get by partial differentiation with respect to x and y,

$$f_x \equiv 2(x + \alpha y + \beta) + 2(1 + \alpha^2)zp = 0, \tag{19}$$

$$f_y \equiv 2\alpha(x + \alpha y + \beta) + 2(1 + \alpha^2)zq = 0, \tag{20}$$

$$g_x \equiv 2(x - a) + 2zp = 0, \tag{21}$$

$$g_y \equiv 2(y - b) + 2zq = 0. \tag{22}$$

From (19) and (21),

$$-zp = \frac{x + \alpha y + \beta}{1 + \alpha^2} = x - a, \tag{23}$$

and from (20) and (22),

$$-zq = \frac{\alpha(x + \alpha y + \beta)}{1 + \alpha^2} = y - b. \tag{24}$$

These two equations combined yield

$$\alpha(x - a) = y - b. \tag{25}$$

From (17) and (18),

$$(1 + \alpha^2)(1 - z^2) = (x + \alpha y + \beta)^2 = (1 + \alpha^2)[(x - a)^2 + (y - b)^2]. \tag{26}$$

Substituting the value of y given by (25) in equation (26), we find

$$(x + \alpha^2 x - a\alpha^2 + b\alpha + \beta)^2 = (1 + \alpha^2)^2(x - a)^2.$$

Hence

$$(1 + \alpha^2)x - a\alpha^2 + b\alpha + \beta = (1 + \alpha^2)(x - a), \tag{27}$$

the sign of the extracted square root being determined by the fact that $(x + \alpha y + \beta)$ must have the same sign as $(x - a)$, by (23). In (27), x cancels, and we are left with the relation

$$\beta = -(a + b\alpha). \tag{28}$$

This is the needed eliminant.

Combining (28) and (17), we then have the one-parameter family of cylinders

$$(x + \alpha y - a - b\alpha)^2 + (1 + \alpha^2)(z^2 - 1) = 0,$$

or

$$(x - a)^2 + 2\alpha(x - a)(y - b) + \alpha^2[(y - b)^2 + z^2 - 1] + z^2 - 1 = 0. \tag{29}$$

Partial differentiation of (29) with respect to α yields

$$\alpha = -\frac{(x - a)(y - b)}{(y - b)^2 + z^2 - 1}. \tag{30}$$

Substituting this expression for α in (29), we get

$$(x - a)^2 + z^2 - 1 - \frac{2(x - a)^2(y - b)^2}{(y - b)^2 + z^2 - 1} + \frac{(x - a)^2(y - b)^2}{(y - b)^2 + z^2 - 1} = 0,$$

$$[(x - a)^2 + (z^2 - 1)][(y - b)^2 + (z^2 - 1)] - (x - a)^2(y - b)^2 = 0,$$

$$(x - a)^2(z^2 - 1) + (y - b)^2(z^2 - 1) + (z^2 - 1)^2 = 0.$$

Factoring out $z^2 - 1$ (which, set equal to zero, incidentally yields the singular integrals), we obtain

$$(x - a)^2 + (y - b)^2 + z^2 - 1 = 0,$$

the second complete integral (18).

EXERCISES

1. In Example 1 of Art. 53, show that taking $\beta = \phi(\alpha)$ will not yield a solution whose characteristic curves are parallel to the x-axis. By setting $\alpha = \phi(\beta)$, obtain the cylinder $12z = y^2$ as the suitable solution.

2. Find the surface satisfying the equation $z = xp + yq + p^2 - q^2$, and generated by characteristic curves parallel to a line with direction numbers 2, 1, 3.

3. Show that the equation $z = xp + yq + 3p - 5q + 2$ has a complete integral representing planes through the point $(-3, 5, 2)$. What planes through this point are not included in the complete integral? Show also that the singular integral is a part of the complete integral.

4. Show that the equation $p^2 + q^2 = z^2$ has the complete integral $z = \beta e^{x \cos \alpha + y \sin \alpha}$ and that the differential equation also possesses a singular integral which is a part of the complete integral.

5. Find a solution of the equation $z = xp + yq + e^{pq-1}$, enveloping planes passing through the point (0, 0, 1) and belonging to a complete integral of the given equation.

6. Find a two-parameter family of planes satisfying the differential equation $z = xp + yq + k\sqrt{p^2 + q^2 + 1}$, and determine the geometric property common to these planes. Hence deduce, on geometric grounds, a singular integral of the differential equation, and find its equation.

7. Find the surface satisfying the differential equation $2q^2 - p + 1 = 0$, and generated by characteristic curves passing through the point (0, 1, 0). Interpret geometrically.

8. Find two surfaces satisfying the differential equation $z = xp + yq - 4p^2 - q$, passing through the point (5, 2, 1), and with characteristic curves parallel to the y-axis.

9. Show that the differential equation $4xzq^2 + p = 0$ possesses the distinct complete integrals

$$z^2 = \alpha y - \alpha^2 x^2 + \beta, \qquad z^2(4x^2 + a) = (y + b)^2.$$

Using the method of Example 2, Art. 53, find a functional relation between α, β, a, and b, and hence find the second solution as a particular case of the general integral obtained from the first.

10. Show that the differential equation $4xzq^2 - p + q = 0$ possesses the distinct complete integrals

$$z^2 = \alpha(x + y) + \alpha^2 x^2 + \beta, \qquad z^2(a - 4x^2) = (x + y + b)^2.$$

Find a functional relation between α, β, a, and b, and hence derive the second solution from the first.

11. Show that the differential equation $z = x_1p_1 + x_2p_2 + x_3p_3 - p_1^2 - 2p_2^2 - 3p_3^2$ has the singular integral $24z = 6x_1^2 + 3x_2^2 + 2x_3^2$, which may be regarded as a hypersurface enveloping the hyperplanes of a complete integral.

12. Show that, for an equation of the form $F(x, y, p, q) = 0$, Charpit's and Jacobi's subsidiary equations are of the same form.

13. Show that a differential equation of the form $F(x_1, x_2, x_3, p_1, p_2, p_3) = 0$ cannot possess a singular integral.

14. Using the method of Art. 50 (*e*), find surfaces whose normals all pass through the line $x = y$, $z = 0$.

15. Find surfaces whose normals all pass through the parabola $y^2 = x$, $z = 0$.

16. Find a non-planar surface for which the tangent planes have intercepts, on the coordinate axes, whose sum is unity. (Cf. Exercise 12, Art. 29.)

17. Find a non-planar surface for which the tangent planes have intercepts, on the coordinate axes, whose product is unity. (Cf. Exercise 13, Art. 29.)

18. Find a non-planar surface for which the tangent planes have intercepts, on the coordinate axes, the sum of whose squares is unity. (Cf. Exercise 14, Art. 29.)

19. Find a non-planar surface for which the product of the distances from the points (1, 0, 0) and (−1, 0, 0) to any tangent plane of the surface is unity.

20. Show that every surface, such that the sum of the intercepts of any tangent plane on the coordinate axes is zero, is a developable surface.

CHAPTER VI

FOURIER SERIES

In the next chapter, concerned with linear partial differential equations of order higher than the first, we shall, sometimes of practical necessity, be led to a consideration of solutions expressed in the form of infinite series involving trigonometric functions. In order that we shall be able to obtain such series as needed, and to acquaint ourselves with the essential properties of these series, we interrupt our study of partial differential equations at this point for a brief discussion of the subject of Fourier trigonometric series.

As intimated above, expression of a given function in the form of a Fourier series rather than in finite or power series form will be required in our process of solving certain partial differential equations. In addition, we shall find that many functions, for each of which there exists no one simple finite expression over its interval of definition, and for which no power series expansion is obtainable, can be treated by the Fourier method.

54. Auxiliary formulas. In our discussion of Fourier series, we shall require the values of certain trigonometric definite integrals. We therefore begin by establishing these integral formulas, to which we can refer as the need arises.

Let c be any number—positive, negative, or zero—and let n denote a positive integer. Then we have, from calculus,

$$\text{(I)} \quad \int_c^{c+2\pi} \sin nx\, dx = \left[-\frac{\cos nx}{n} \right]_c^{c+2\pi}$$

$$= -\frac{1}{n}\left[\cos(nc + 2\pi n) - \cos nc\right] = 0,$$

since $\cos(\theta + 2\pi n) \equiv \cos\theta$ for n an integer, and similarly,

$$\text{(II)} \qquad \int_c^{c+2\pi} \cos nx\, dx = \left[\frac{\sin nx}{n}\right]_c^{c+2\pi} = 0.$$

Further useful formulas may be obtained by using the trigonometric identities

$$\sin mx \cos nx = \tfrac{1}{2}[\sin (m-n)x + \sin (m+n)x],$$
$$\sin mx \sin nx = \tfrac{1}{2}[\cos (m-n)x - \cos (m+n)x],$$
$$\cos mx \cos nx = \tfrac{1}{2}[\cos (m-n)x + \cos (m+n)x].$$

With m, as well as n, a positive integer, we then get with the aid of the above identities and formulas (I) and (II),

$$\text{(III)} \qquad \int_c^{c+2\pi} \sin mx \cos nx \, dx = 0;$$

and, with $m \neq n$,

$$\text{(IV)} \qquad \int_c^{c+2\pi} \sin mx \sin nx \, dx = 0, \qquad (m \neq n),$$

$$\text{(V)} \qquad \int_c^{c+2\pi} \cos mx \cos nx \, dx = 0, \qquad (m \neq n).$$

Finally, we have

$$\text{(VI)} \qquad \int_c^{c+2\pi} \sin^2 nx \, dx = \tfrac{1}{2}\int_c^{c+2\pi} (1 - \cos 2nx)\, dx = \pi,$$

$$\text{(VII)} \qquad \int_c^{c+2\pi} \cos^2 nx \, dx = \tfrac{1}{2}\int_c^{c+2\pi} (1 + \cos 2nx)\, dx = \pi.$$

These seven integral formulas are the ones required.

55. Definitions and theorems. A *trigonometric series* is defined as a series of the form

$$\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + \cdots$$
$$+ b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx + \cdots, \tag{1}$$

where the a's and b's are constants and x is a variable. Later we shall obtain a slightly different form of trigonometric series, in which the variable is replaced by some simple linear function of x, but such a form is essentially equivalent to (1).

For convenience, we sometimes represent the series (1) in the compact form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{2}$$

Similar summation notations will occasionally be used for other series in the following discussion.

The constant term in (1) is written as $a_0/2$ rather than as a_0 for subsequent convenience, for we shall find a formula for a_n which is valid when $n = 0$ as well as when n is a positive integer.

It will be remembered that $\sin nx$ and $\cos nx$ have the period $2\pi/n$, and that any integral multiple of this number is also a period; that is, for k an integer,

$$\sin n\left(x + \frac{2\pi k}{n}\right) \equiv \sin (nx + 2\pi k) \equiv \sin nx,$$

$$\cos n\left(x + \frac{2\pi k}{n}\right) \equiv \cos (nx + 2\pi k) \equiv \cos nx.$$

Thus every term in the Fourier series (1) has 2π as a period, and consequently any function represented by a series of the form (1) in an interval of length 2π will be periodic, with period 2π, and automatically defined by the series (1) for every x. We therefore confine ourselves (temporarily) to intervals of length 2π; such an interval may have any left-hand boundary value c, and will then extend to $c + 2\pi$.

The conditions under which a series of the form (1) will converge and represent a function $f(x)$ in an interval of length 2π cannot be discussed here.* We shall merely determine, under adequate assumptions, values of the coefficients a_0, a_n, and b_n $(n = 1, 2, \cdots)$ corresponding to a given suitable function $f(x)$, and shall then state Fourier's theorem.

Suppose that a given function $f(x)$ is expressible by a trigonometric series,

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx + \cdots, \quad (3)$$

and suppose further that this series may be integrated term by term. Then we have, using formulas (I) and (II) of Art. 54,

$$\int_c^{c+2\pi} f(x)\, dx = \int_c^{c+2\pi} \frac{a_0}{2}\, dx = \pi a_0,$$

whence

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\, dx. \quad (4)$$

* See, for example, Goursat-Hedrick, "Mathematical Analysis," vol. I.

Now multiply both members of (3) by $\cos nx$, and again integrate term by term. With the aid of formulas (II), (III), (V), and (VII) of Art. 54, we then find

$$\int_c^{c+2\pi} f(x) \cos nx \, dx = \int_c^{c+2\pi} a_n \cos^2 nx \, dx = \pi a_n,$$

so that for $n = 1, 2, \cdots$,

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \cos nx \, dx. \tag{5}$$

We note that relation (5) also holds for $n = 0$, for then it reduces to (4); it is for this reason that the constant term in our series is conventionally denoted by $a_0/2$.

Finally, multiply each member of (3) by $\sin nx$, and integrate term by term. Making use of formulas (I), (III), (IV), and (VI) of Art. 54, the result is

$$\int_c^{c+2\pi} f(x) \sin nx \, dx = \int_c^{c+2\pi} b_n \sin^2 nx \, dx = \pi b_n,$$

and therefore, for $n = 1, 2, \cdots$,

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \sin nx \, dx. \tag{6}$$

Thus, given a function $f(x)$, we can formally compute numbers a_0, a_n, b_n ($n = 1, 2, \cdots$) by means of relations (4)–(6). The trigonometric series having coefficients obtained in this manner is called the *Fourier series* belonging to the function $f(x)$, and we call the numbers a_0, a_n, b_n ($n = 1, 2, \cdots$) the *Fourier coefficients* of the function $f(x)$. Relations (4)–(6) may be briefly referred to as *Fourier formulas*.

The student who is familiar with the concept of uniform convergence will recognize the fact that our assumptions are fulfilled when the series (1) converges uniformly, and represents the function $f(x)$, in the closed interval $c \leq x \leq c + 2\pi$. We thus state the following theorem.

THEOREM I. *If the series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

is uniformly convergent in the interval $c \leqq x \leqq c + 2\pi$, *and represents the function* $f(x)$ *in that interval, then*

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \cos nx\, dx, \qquad (n = 0, 1, 2, \cdots),$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \sin nx\, dx, \qquad (n = 1, 2, \cdots).$$

Now it is not obvious that every function $f(x)$ will be representable by its Fourier series, formally constructed by means of relations (4)–(6), nor, in fact, that the Fourier series will converge to any function. It actually turns out that the Fourier series belonging to certain functions do not converge to any function. However, all the functions that normally arise in practice, including many possessing no power series

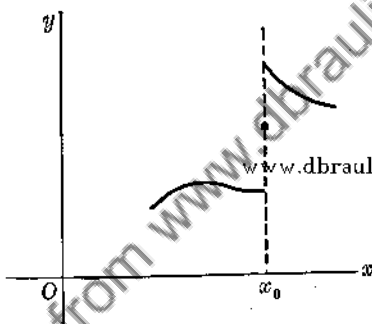

FIG. 12

expansions, are properly represented by Fourier series. In this connection we state *Fourier's theorem:*

THEOREM II. *Any single-valued function* $f(x)$, *defined and continuous except possibly for a finite number of finite discontinuities in an interval of length* 2π, *and having only a finite number of maxima and minima in that interval, possesses a convergent Fourier series which represents it.*

A function $f(x)$ is said to have a finite discontinuity at a point $x = x_0$ if $\lim_{h \to 0^+} f(x_0 - h)$ and $\lim_{h \to 0^+} f(x_0 + h)$ both exist but are different numbers. Thus a finite discontinuity is represented geometrically by a finite gap or jump in the graph of the function, as shown in Fig. 12. It turns out that, when we put $x = x_0$ in a Fourier series for a function having a finite discontinuity at this point, the series yields the arithmetic mean of the two foregoing limits.

56. Examples. We now illustrate, by means of examples, the manner in which Fourier series belonging to various functions are constructed. Some of our results will also be of use in later work.

Example 1. Expand the function defined by the relations

$$\begin{aligned} f(x) &= -1, \qquad -\pi < x < 0, \\ f(x) &= 1, \qquad\quad 0 < x < \pi, \end{aligned} \tag{1}$$

in a Fourier series.

Solution. Here the interval, of length 2π, begins at $c = -\pi$. The function, whose graph is shown in Fig. 13, is in this case defined through two relations, and has a finite discontinuity at $x = 0$. According to the statement at the end

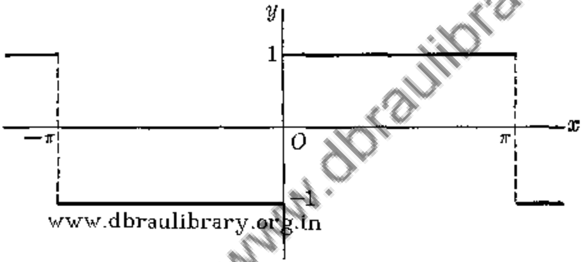

FIG. 13

of Art. 55, the Fourier series should yield $\frac{1}{2}(1 - 1) = 0$ for $x = 0$. In addition, the Fourier series will, because of its periodicity, define $f(x)$ for every x, and there will be finite discontinuities at $x = \pm\pi, \pm 2\pi, \cdots$, where the series should have the sum 0.

From the Fourier formulas of Art. 55, we get

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-1)\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\cdot dx = -1 + 1 = 0,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\cdot\cos nx\,dx$$

$$= \left[-\frac{\sin nx}{n\pi}\right]_{-\pi}^{0} + \left[\frac{\sin nx}{n\pi}\right]_{0}^{\pi} = 0, \qquad (n = 1, 2, \cdots),$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\cdot\sin nx\,dx$$

$$= \left[\frac{\cos nx}{n\pi}\right]_{-\pi}^{0} + \left[-\frac{\cos nx}{n\pi}\right]_{0}^{\pi} = \frac{1}{n\pi}\left[1 - \cos(-n\pi) - \cos n\pi + 1\right]$$

$$= \frac{2}{n\pi}(1 - \cos n\pi), \qquad (n = 1, 2, \cdots).$$

Thus, all a's vanish, and, since the cosine of an even multiple of π is 1, all b's with even subscript also vanish. For $n = 1, 3, 5, \cdots$, we get

$$b_1 = \frac{4}{\pi}, \qquad b_3 = \frac{4}{3\pi}, \qquad b_5 = \frac{4}{5\pi}, \cdots.$$

Therefore the required Fourier series, for $-\pi < x < \pi$, is

$$f(x) = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right). \tag{2}$$

Evidently this series yields $f(0) = f(\pm\pi) = f(\pm 2\pi) = \cdots = 0$, as we expected.

Example 2. Find the Fourier expansion of the function defined as

$$\begin{aligned} f(x) &= x, & 0 < x \leqq \pi, \\ f(x) &= 2\pi - x, & \pi < x < 2\pi. \end{aligned} \tag{3}$$

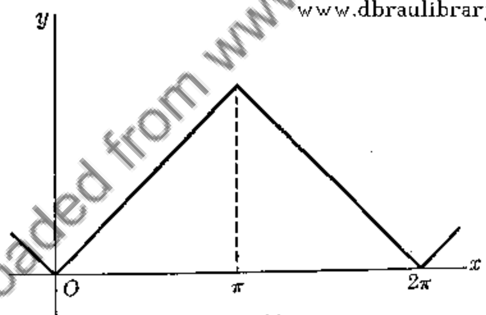

FIG. 14

Solution. This function, which is continuous throughout the interval $0 < x < 2\pi$, has the graph shown in Fig. 14. We now find, with $c = 0$,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2\pi - x)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} + \frac{1}{\pi}\left[-\frac{(2\pi - x)^2}{2}\right]_{\pi}^{2\pi} = \frac{\pi}{2} + \frac{\pi}{2} = \pi,$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^\pi + \frac{1}{\pi}\left[\frac{2\pi \sin nx}{n} - \frac{x \sin nx}{n} - \frac{\cos nx}{n^2}\right]_\pi^{2\pi}$$

$$= \frac{\cos n\pi - 1}{n^2\pi} + \frac{-1 + \cos n\pi}{n^2\pi} = -\frac{2}{n^2\pi}(1 - \cos n\pi), \qquad (n = 1, 2, \cdots),$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx = \frac{1}{\pi}\int_0^\pi x \sin nx\, dx + \frac{1}{\pi}\int_\pi^{2\pi} (2\pi - x) \sin nx\, dx$$

$$= \frac{1}{\pi}\left[-\frac{x \cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^\pi + \frac{1}{\pi}\left[-\frac{2\pi \cos nx}{n} + \frac{x \cos nx}{n} - \frac{\sin nx}{n^2}\right]_\pi^{2\pi}$$

$$= -\frac{\cos n\pi}{n} - \frac{2}{n} + \frac{2}{n} + \frac{2 \cos n\pi}{n} - \frac{\cos n\pi}{n} = 0, \qquad (n = 1, 2, \cdots).$$

Thus all of the b's, and the a's with even subscript, vanish, and

$$a_0 = \pi, \qquad a_1 = -\frac{4}{\pi}, \qquad a_3 = -\frac{4}{3^2\pi}, \qquad a_5 = -\frac{4}{5^2\pi}, \qquad \cdots.$$

Therefore the desired expansion, for $0 < x < 2\pi$, is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right). \tag{4}$$

Example 3. Find the Fourier series for the function

$$f(x) = x, \qquad 0 < x \leqq \pi,$$
$$f(x) = \pi, \qquad \pi < x < 2\pi. \tag{5}$$

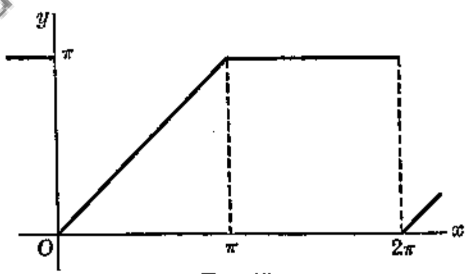

FIG. 15

Solution. This function, continuous for $0 < x < 2\pi$, coincides with the function of Example 2 in the left-hand half of the interval, but is different in the right-hand half (Fig. 15). Here there are finite discontinuities at the end points of the interval, which was not the case in Example 2.

The Fourier coefficients now are

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\,dx = \frac{\pi}{2} + \pi = \frac{3\pi}{2},$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\cos nx\,dx$$

$$= \frac{\cos n\pi - 1}{n^2\pi} + 0 = -\frac{1}{n^2\pi}(1 - \cos n\pi), \qquad (n = 1, 2, \cdots),$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\sin nx\,dx$$

$$= -\frac{\cos n\pi}{n} - \frac{1}{n} + \frac{\cos n\pi}{n} = -\frac{1}{n}, \qquad (n = 1, 2, \cdots).$$

Therefore, for $0 < x < 2\pi$,

$$f(x) = \frac{3\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right)$$
$$- (\sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x + \cdots). \qquad (6)$$

EXERCISES

1. Show that the function defined as $f(x) = \sin 1/x$, $0 < x < 2\pi$, does not satisfy the conditions of Fourier's theorem.

2. Setting $x = \pi/4$ and $x = \pi/2$ in equation (2), Art. 56, deduce two numerical series for π.

3. Verify series (4), Art. 56, for $x = \pi/2$. Also, setting $x = 0$ in (4), deduce a numerical series for π^2.

4. Setting $x = \pi/2$ and $x = \pi$ in equation (6), Art. 56, deduce series for π and π^2. Using the latter result, obtain the value of $f(0)$ as given by (6).

In Exercises 5–10, expand each function in a Fourier series over the given interval. Also examine each series at points of discontinuity when such exist.

5. $f(x) = 1$, $0 < x < \pi$; $f(x) = 0$, $\pi < x < 2\pi$.

6. $f(x) = 0$, $-\pi < x < 0$; $f(x) = x$, $0 < x < \pi$.

7. $f(x) = x$, $0 < x \leqq \pi/2$; $f(x) = \pi/2$, $\pi/2 < x \leqq 3\pi/2$; $f(x) = 2\pi - x$, $3\pi/2 < x < 2\pi$.

8. $f(x) = \sin x$, $-\pi < x \leqq 0$; $f(x) = 0$, $0 < x < \pi$.

9. $f(x) = \sin x/2$, $-\pi < x < \pi$.

10. $f(x) = |\sin x|$, $0 < x < 2\pi$.

57. Even and odd functions. A function which is such that $f(-x) \equiv f(x)$ is called an *even function*. For example, $\cos x$ is an even function, since $\cos(-x) \equiv \cos x$. The geometric characteristic of the graph of an even function, $y = f(x)$, is symmetry with respect to the y-axis; thus, the graph (Fig. 14) of the function of Example 2, Art. 56, shows that this function is even.

An *odd function* is a function $f(x)$ such that $f(-x) \equiv -f(x)$. Thus, $\sin x$ is odd, for $\sin(-x) \equiv -\sin x$. The graph of an odd function is symmetric with respect to the origin; Fig. 13 shows that the function of Example 1, Art. 56, is odd.

Now it was found in Example 1, Art. 56, that the Fourier coefficients a_0, a_n $(n = 1, 2, \cdots)$, of that odd function, all vanished; and the coefficients b_n of the even function of Example 2, Art. 56, were likewise all zero. These facts could have been predicted; in this connection we have the following theorem.

THEOREM. *The Fourier coefficients of an even function $f(x)$ expanded over the interval from $-\pi$ to π are given by*

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx, \qquad b_n = 0. \tag{1}$$

The Fourier coefficients of an odd function $f(x)$ for the same interval are given by

$$a_n = 0, \qquad b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx. \tag{2}$$

Geometric consideration of the areas represented by the integrals in the Fourier formulas readily shows this theorem to be valid. Here we shall prove analytically only the first half of the theorem. We have

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_0^\pi f(x)\cos nx\,dx. \end{aligned}$$

In the first integral of the latter expression, change x into $-x$; remembering that $\cos(-nx) \equiv \cos nx$, and that $f(-x) \equiv f(x)$ by supposition, we get

$$a_n = \frac{1}{\pi}\int_{\pi}^{0} f(x)\cos nx\,(-dx) + \frac{1}{\pi}\int_0^\pi f(x)\cos nx\,dx.$$

Now change the order of integration in the first integral. This changes the sign of that integral, and it becomes a duplicate of the second integral, whence a_n is seen to be given by (1). Moreover,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x) \sin nx \, dx + \frac{1}{\pi}\int_{0}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{\pi}^{0} f(x) \sin (-nx)\, (-dx) + \frac{1}{\pi}\int_{0}^{\pi} f(x) \sin nx \, dx$$

$$= -\frac{1}{\pi}\int_{0}^{\pi} f(x) \sin nx \, dx + \frac{1}{\pi}\int_{0}^{\pi} f(x) \sin nx \, dx = 0.$$

Consequently the second of equations (1) is established. Formulas (2) may be proved in like fashion.

When a given function, to be expanded in a Fourier series over the interval from $-\pi$ to π, is either even or odd, the above theorem effects a considerable reduction in the computation. If the function is neither even nor odd (for instance, the function of Example 3, Art. 56), or if the interval of definition is not $-\pi < x < \pi$ (or one equivalent to it, as $\pi < x < 3\pi$), we use the original Fourier formulas of Art. 55.

Example. Obtain the Fourier series for the function

$$f(x) = x, \qquad -\pi < x < \pi. \tag{3}$$

Solution. Since this function is odd, the theorem tells us that all the a's will vanish, and we need compute only the b's, using the second of equations (2). We get

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} x \sin nx \, dx = \frac{2}{\pi}\left[-\frac{x \cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} = -\frac{2 \cos n\pi}{n}.$$

Therefore, for $-\pi < x < \pi$,

$$x = 2(\sin x - \tfrac{1}{2} \sin 2x + \tfrac{1}{3} \sin 3x - \tfrac{1}{4} \sin 4x + \cdots). \tag{4}$$

The graph of this function is shown in Fig. 16. We may check by noting that series (4) yields $f(0) = f(\pm\pi) = 0$, as expected.

58. Half-range series. Let there be given a function $f(x)$ defined for $0 < x < \pi$, an interval of only half the length previously considered. For some purposes, it may become necessary to expand this function in a series of sines alone or in a series of cosines alone. We shall find instances of these needs in various problems in the next chapter.

The properties of even functions and odd functions, discussed in the preceding article, enable us to fulfill such requirements. Suppose first that a series consisting only of sine terms is needed. Since this sort of

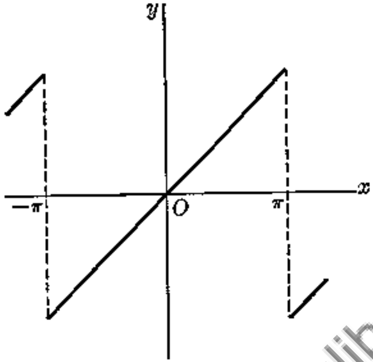

FIG. 16

series arises in connection with odd functions, we create an odd function $F_0(x)$, defined for the full range, $-\pi < x < \pi$, and which is identical with $f(x)$ in the right-hand half-range, $0 < x < \pi$. It is easy to see that this odd function $F_0(x)$ will be given by

$$\begin{aligned} F_0(x) &= -f(-x), \qquad -\pi < x < 0, \\ F_0(x) &= f(x), \qquad 0 < x < \pi. \end{aligned} \tag{1}$$

The Fourier series for $F_0(x)$ will then consist only of sine terms, and, since $F_0(x) = f(x)$ for $0 < x < \pi$, this sine series will represent the given function $f(x)$ in its half-range.

If, on the other hand, we need a cosine series representing the given function $f(x)$ for $0 < x < \pi$, we proceed as follows. We create an even function $F_E(x)$, given by

$$\begin{aligned} F_E(x) &= f(-x), \qquad -\pi < x < 0, \\ F_E(x) &= f(x), \qquad 0 < x < \pi. \end{aligned} \tag{2}$$

Then, by the theorem of Art. 57, the Fourier series for $F_E(x)$ will contain only cosine terms, and, by the second of equations (2), this cosine series will represent the given function $f(x)$ in its half-range.

Usually we are entirely unconcerned with the yield of either one of these half-range series in the left-hand half, $-\pi < x < 0$, for such a yield is merely a by-product arising from the artificially created function, and our interest is centered in the representation of $f(x)$ for $0 < x < \pi$.

Accordingly, we need not stop to construct the function $F_0(x)$ or $F_E(x)$, as the case may be, but may directly apply the theorem of Art. 57 to the function $f(x)$, as in the following example.

Example. Find the half-range sine expansion and the half-range cosine expansion of the function

$$f(x) = x - 1, \qquad 0 < x < \pi. \tag{3}$$

Solution. To find the sine series, we use formulas (2) of Art. 57. We get, of course, $a_n = 0$, and

$$b_n = \frac{2}{\pi}\int_0^\pi (x - 1)\sin nx\,dx = \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2} + \frac{\cos nx}{n}\right]_0^\pi$$

$$= -\frac{2\cos n\pi}{n} + \frac{2\cos n\pi}{n\pi} - \frac{2}{n\pi} = \frac{2}{n\pi}(\cos n\pi - \pi\cos n\pi - 1).$$

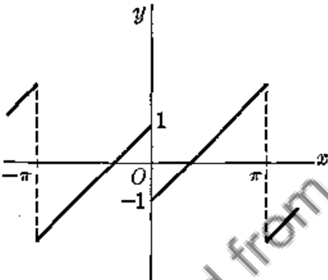

FIG. 17

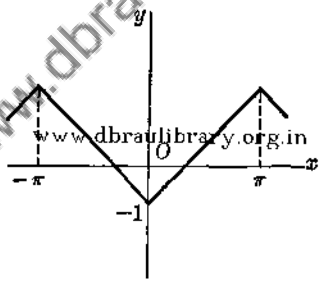

FIG. 18

Therefore, for $0 < x < \pi$,

$$f(x) = \frac{2}{\pi}\left[(\pi - 2)\sin x - \frac{\pi}{2}\sin 2x + \frac{\pi - 2}{3}\sin 3x - \frac{\pi}{4}\sin 4x + \cdots\right]. \tag{4}$$

Similarly, formulas (1) of Art. 57 give us the coefficients of the cosine series. We find, in addition to $b_n = 0$,

$$a_0 = \frac{2}{\pi}\int_0^\pi (x - 1)\,dx = \frac{2}{\pi}\left[\frac{(x-1)^2}{2}\right]_0^\pi = \frac{(\pi - 1)^2 - 1}{\pi} = \pi - 2,$$

$$a_n = \frac{2}{\pi}\int_0^\pi (x - 1)\cos nx\,dx = \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2} - \frac{\sin nx}{n}\right]_0^\pi$$

$$= \frac{2\cos n\pi}{n^2\pi} - \frac{2}{n^2\pi} = -\frac{2}{n^2\pi}(1 - \cos n\pi).$$

Hence, for $0 < x < \pi$,

$$f(x) = \frac{\pi - 2}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right). \tag{5}$$

The graphs of the functions given by series (4) and (5) are shown in Figs. 17 and 18 respectively. The differences in these two series are accounted for by the difference in the graphs for $-\pi < x < 0$. However, the graphs coincide for $0 < x < \pi$, as they should.

EXERCISES

1. Show that any function defined for every value of x may be expressed as the sum of an even function and an odd function.

2. Prove analytically the second half of the theorem of Art. 57.

3. Deduce the theorem of Art. 57 on geometric grounds.

4. Setting $x = \pi/2$ in series (4), Art. 57, deduce a series for π. Using this result, check series (4) of Art. 58 for $x = \pi/2$.

5. Setting $x = 0$ in series (5), Art. 58, deduce a series for π^2. Using this result, check the series for $x = \pi$.

In Exercises 6–10, find both sine and cosine half-range series for each of the given functions.

6. $f(x) = 0, \quad 0 < x < \pi/2; \quad f(x) = 1, \quad \pi/2 < x < \pi.$

7. $f(x) = 1, \quad 0 < x < \pi/2; \quad f(x) = -1, \quad \pi/2 < x < \pi.$

8. $f(x) = x, \quad 0 < x \leqq \pi/2; \quad f(x) = \pi/2, \quad \pi/2 < x < \pi.$

9. $f(x) = x^2, \ 0 < x < \pi.$

10. $f(x) = e^{-x}, 0 < x < \pi.$

59. Change of interval. In many problems involving trigonometric series, functions must be expanded over intervals of length different from 2π or π. It therefore becomes desirable to develop methods and formulas which will enable us to expand a function defined over any interval.

Let a function $f(x)$ be defined for $-L < x < L$, say, where L is any positive number. One way of attacking our problem is to get an expansion of the function for the range $-\pi < x < \pi$, and then transform the latter interval into the desired one, from $-L$ to L, by means of a suitable substitution. In effect, such a substitution will elongate, or compress, the interval from $-\pi$ to π into the interval from $-L$ to L; it is equivalent to a change in the unit of length, the ratio of these units being π/L.

Evidently the linear transformation

$$z = \frac{\pi x}{L} \tag{1}$$

will effect the desired change of scale, for as x varies from $-L$ to L, z will vary from $-\pi$ to π. Now let $f(x) = f(Lz/\pi) = g(z)$, say, be developed in a Fourier series,

$$g(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nz + b_n \sin nz), \tag{2}$$

valid for $-\pi < z < \pi$. Under the transformation (1), this becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right), \tag{3}$$

valid for $-L < x < L$. This series (3) will then be a suitable expansion.

We may obtain formulas for the a's and b's in (3) directly by applying transformation (1) to the Fourier formulas already developed. We have

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \cos nz\, dz = \frac{1}{\pi} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, d\left(\frac{\pi x}{L}\right)$$

$$= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx, \tag{4}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(z) \sin nz\, dz = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx. \tag{5}$$

Formulas (4) and (5) enable us to derive the series (3) in each particular case without going through the transformation process.

Half-range sine or cosine series for the interval $0 < x < L$ may be similarly found. We state all essential matters in the form of a theorem. It is, of course, assumed that the function $f(x)$ under consideration satisfies suitable conditions, such as those of Theorem II, Art. 55, for the interval involved.

THEOREM. *A function $f(x)$, defined over an interval $-L < x < L$, has the Fourier expansion*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx, \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx.$$

A function $f(x)$, defined over an interval $0 < x < L$, has the Fourier half-range sine expansion

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \qquad (n = 1, 2, \cdots);$$

and it has the Fourier half-range cosine expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L},$$

where

$$a_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi x}{L}\, dx, \qquad (n = 0, 1, 2, \cdots).$$

Example. Expand the function

$$f(x) = x^2, \qquad 0 < x < L, \tag{6}$$

in a Fourier half-range cosine series.

Solution. From the above theorem, we get

$$a_0 = \frac{2}{L}\int_0^L x^2\, dx = \frac{2L^2}{3},$$

$$a_n = \frac{2}{L}\int_0^L x^2 \cos \frac{n\pi x}{L}\, dx = \frac{2}{L}\left[\frac{2L^2 x}{n^2\pi^2}\cos\frac{n\pi x}{L} + \frac{Lx^2}{n\pi}\sin\frac{n\pi x}{L} - \frac{2L^3}{n^3\pi^3}\sin\frac{n\pi x}{L}\right]_0^L$$

$$= \frac{4L^2}{n^2\pi^2}\cos n\pi.$$

Therefore, for $0 < x < L$,

$$x^2 = \frac{L^2}{3} - \frac{4L^2}{\pi^2}\left(\cos\frac{\pi x}{L} - \frac{1}{2^2}\cos\frac{2\pi x}{L} + \frac{1}{3^2}\cos\frac{3\pi x}{L} - \cdots\right). \tag{7}$$

Inasmuch as the function x^2 is even, series (7) incidentally represents this function throughout the full range, $-L < x < L$.

60. Transformation and combination of series. The scale transformation (1) of Art. 59 may be used, as described, to convert previously obtained series into other useful series. Because of their utility in later work, we now derive five important series from earlier results.

From equation (2) of Art. 56, we get the development of unity in a half-range sine series.

$$1 = \frac{4}{\pi}\left(\sin z + \tfrac{1}{3}\sin 3z + \tfrac{1}{5}\sin 5z + \cdots\right), \qquad 0 < z < \pi.$$

Using the transformation

$$z = \frac{\pi x}{L},$$

where L is any positive number, we find

$$\text{(I)} \qquad 1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} + \frac{1}{5}\sin\frac{5\pi x}{L} + \cdots\right), \quad 0 < x < L.$$

This is the first of our five basic series.

There is no infinite cosine series for unity. For, $f(x) = 1$, $-\pi < x < \pi$, is an even function, and a Fourier series should yield $f(x) = 1$ for every x. The Fourier process then leads to $a_0 = 2$, $a_n = 0$ $(n = 1, 2, \cdots)$, whence we get the triviality $1 = 1$.

From equation (4) of Art. 57, we get the half-range sine series for z,

$$z = 2(\sin z - \tfrac{1}{2}\sin 2z + \tfrac{1}{3}\sin 3z - \cdots), \qquad 0 < z < \pi.$$

Changing z into $\pi x/L$, this yields our second basic series,

$$\text{(II)} \qquad x = \frac{2L}{\pi}\left(\sin\frac{\pi x}{L} - \frac{1}{2}\sin\frac{2\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L} - \cdots\right), \quad 0 < x < L.$$

Equation (4) of Art. 56 gives us the half-range cosine series for z,

$$z = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos z + \frac{1}{3^2}\cos 3z + \frac{1}{5^2}\cos 5z + \cdots\right), \qquad 0 < z < \pi.$$

Setting $z = \pi x/L$, this leads to

$$\text{(III)} \qquad x = \frac{L}{2} - \frac{4L}{\pi^2}\left(\cos\frac{\pi x}{L} + \frac{1}{3^2}\cos\frac{3\pi x}{L} + \frac{1}{5^2}\cos\frac{5\pi x}{L} + \cdots\right),$$
$$0 < x < L.$$

Thus, (II) and (III) are two series for x over the general half-range, $0 < x < L$.

Equation (7) of Art. 59 gives us directly

$$\text{(IV)} \qquad x^2 = \frac{L^2}{3} - \frac{4L^2}{\pi^2}\left(\cos\frac{\pi x}{L} - \frac{1}{2^2}\cos\frac{2\pi x}{L} + \frac{1}{3^2}\cos\frac{3\pi x}{L} - \cdots\right),$$

$$0 < x < L.$$

Finally, either from the sine series for z^2, $0 < z < \pi$ (cf. Art. 58, Exercise 9), or from the theorem of Art. 59, there may be obtained the series

$$\text{(V)} \qquad x^2 = \frac{2L^2}{\pi^3}\left[(\pi^2 - 4)\sin\frac{\pi x}{L} - \frac{\pi^2}{2}\sin\frac{2\pi x}{L} + \left(\frac{\pi^2}{3} - \frac{4}{3^3}\right)\sin\frac{3\pi x}{L} - \frac{\pi^2}{4}\sin\frac{4\pi x}{L} + \left(\frac{\pi^2}{5} - \frac{4}{5^3}\right)\sin\frac{5\pi x}{L} - \frac{\pi^2}{6}\sin\frac{6\pi x}{L} + \cdots\right],$$

$$0 < x < L.$$

Series (IV) and (V) are the basic sine and cosine series for the function x^2 over the general half-range, $0 < x < L$.

The series (I)–(V) may easily be combined to yield Fourier expansions of any linear or quadratic function of x over the general half-range, $0 < x < L$. We illustrate the process by means of an example.

Example. Find a sine series and a cosine series for the function

$$f(x) = 2x - 4, \qquad 0 < x < 4.$$

Solution. To get a sine series for this function, we combine series (I) and (II). Multiplying (II) by 2, and (I) by 4, and subtracting the latter from the former result, while at the same time setting $L = 4$ in both series, we get

$$2x - 4 = \frac{16}{\pi}\left(\sin\frac{\pi x}{4} - \frac{1}{2}\sin\frac{2\pi x}{4} + \frac{1}{3}\sin\frac{3\pi x}{4} - \cdots\right)$$
$$- \frac{16}{\pi}\left(\sin\frac{\pi x}{4} \qquad\qquad + \frac{1}{3}\sin\frac{3\pi x}{4} + \cdots\right).$$

Evidently the terms involving odd multiples of $\pi x/4$ cancel, and the result is, after simple reductions,

$$2x - 4 = -\frac{8}{\pi}\left(\sin\frac{\pi x}{2} + \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \cdots\right), \qquad 0 < x < 4.$$

The cosine series may be similarly found. Multiplying (III) by 2 and subtracting 4, and also setting $L = 4$, we have

$$2x - 4 = -\frac{32}{\pi^2}\left(\cos\frac{\pi x}{4} + \frac{1}{3^2}\cos\frac{3\pi x}{4} + \frac{1}{5^2}\cos\frac{5\pi x}{4} + \cdots\right), \qquad 0 < x < 4.$$

EXERCISES

1. Show that a linear substitution of the form $z = \alpha x + \beta$ will transform any interval $a \leqq z \leqq b$ into any other interval $c \leqq x \leqq d$.

2. Using the proper substitution, transform the sine series for unity, $0 < z < \pi$, into the series

$$\frac{4}{\pi}\left(\cos\frac{\pi x}{2L} - \frac{1}{3}\cos\frac{3\pi x}{2L} + \frac{1}{5}\cos\frac{5\pi x}{2L} - \cdots\right) \qquad -L < x < L.$$

3. Derive series (V) of Art. 60.

4. Expand $f(x) = x - 1, \quad 0 < x < \pi$, in a sine series by combining suitable series from Art. 60. (Cf. Example, Art. 58.)

5. Expand $f(x) = x - 1, \quad 0 < x < \pi$, in a cosine series by combining known series. (Cf. Example, Art. 58.)

6. By combining series, obtain a sine series for $f(x) = 3x - 2, \quad 0 < x < 1$.

7. By combining series, obtain a cosine series for $f(x) = 4x - 3, \quad 0 < x < 2$.

8. Using the theorem of Art. 59, expand the function $f(x) = 2x + 3, \quad -2 < x < 2$.

9. Using the theorem of Art. 59, expand the function $f(x) = 4 - x, \quad -3 < x < 3$.

10. By combining series, obtain a sine expansion of $f(x) = x^2 - 2, \quad 0 < x < 2$.

11. Obtain a cosine expansion of the function of Exercise 10.

12. Derive a sine series for the function $f(x) = 4x - x^2, \quad 0 < x < 4$.

13. Obtain a cosine series for the function of Exercise 12.

14. Derive a sine series for the function $f(x) = (x - 1)^2, \quad 0 < x < 1$.

15. Obtain a cosine series for the function of Exercise 14.

16. Expand the function $f(x) = e^x, \quad 0 < x < 1$, in a sine series.

17. Obtain a cosine series for the function of Exercise 16.

18. Expand the function $f(x) = e^{-x}, \quad 0 < x < 1$, in a sine series.

19. Obtain a cosine series for the function of Exercise 18.

20. From the sine series of Exercises 16 and 18 obtain sine expansions of $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x}), \quad 0 < x < 1$. Similarly, combine the cosine series of Exercises 17 and 19 to get cosine expansions of $\sinh x$ and $\cosh x, \quad 0 < x < 1$.

CHAPTER VII

LINEAR EQUATIONS OF SECOND AND HIGHER ORDERS

In this chapter and the following one, we shall discuss certain types of partial differential equations of order higher than the first. Only the more important forms of higher order equations and the various methods of solving them can properly be considered in a first course, and our treatment of such equations will accordingly be not as complete as it was for first order equations.

One of the reasons necessitating a restricted treatment of higher order equations may be traced to the peculiarities encountered when deriving these equations as eliminants. For instance, it was found in Example 3 of Art. 28 that it was possible to eliminate the six quantities f, g, f', g', f'', and g'' from the six relations at our disposal, and we were thus able to get a second order partial differential equation as the eliminant. In general, however, as stated in Art. 28, the elimination of n arbitrary functions leads to more than one differential equation of order greater than n (cf. Exercise 18, Art. 28). Consequently we should expect that only for certain partial differential equations of the second order can we get solutions each of which contains two arbitrary functions, but that we shall have to be content with less general solutions for other second order equations.

We shall therefore not attempt to classify solutions of partial differential equations of higher orders into categories such as complete, general, etc. Instead, for each type of differential equation considered, we shall find whatever solutions our methods furnish, or those solutions of greatest usefulness in applications.

61. Definitions and a theorem. Because of their importance and tractability, we devote this chapter entirely to linear partial differential equations. For $m \geqq 2$, a linear equation of order m is defined as an equation of the first degree in the dependent variable and its derivatives. This definition is thus similar to that of linear ordinary differential equations (Art. 10), whereas the definition of a linear partial differential equation of first order (Chapter IV) does not stipulate linearity for the dependent variable, but only for its derivatives.

When only two independent variables, x and y, are involved, as will be the case in nearly all our work, the linear partial differential equation may be written in the compact symbolic form (cf. Art. 10)

$$\phi(D_x, D_y)z = F(x, y), \tag{1}$$

where

$$D_x \equiv \frac{\partial}{\partial x}, \quad D_y \equiv \frac{\partial}{\partial y}, \quad D_x^j D_y^k \equiv \frac{\partial^{j+k}}{\partial x^j \partial y^k}, \tag{2}$$

and where the *operator* $\phi(D_x, D_y)$ denotes a polynomial of degree m in D_x and D_y, with coefficients which are functions of x and y only.

When, moreover, the equation (1) is of the second order, we shall frequently employ the notation

$$p \equiv \frac{\partial z}{\partial x}, \quad q \equiv \frac{\partial z}{\partial y}, \quad r \equiv \frac{\partial p}{\partial x} \equiv \frac{\partial^2 z}{\partial x^2}, \quad t \equiv \frac{\partial q}{\partial y} \equiv \frac{\partial^2 z}{\partial y^2},$$

$$s \equiv \frac{\partial p}{\partial y} \equiv \frac{\partial q}{\partial x} \equiv \frac{\partial^2 z}{\partial x \partial y} \equiv \frac{\partial^2 z}{\partial y \partial x}. \tag{3}$$

Thus, the second order linear equation may be written either as

$$(RD_x^2 + SD_xD_y + TD_y^2 + PD_x + QD_y + Z)z = F(x, y), \tag{4}$$

or as

$$Rr + Ss + Tt + Pp + Qq + Zz = F(x, y), \tag{5}$$

where R, S, T, P, Q, and Z, as well as F, are functions of x and y only.

If the right member in (1) is zero, so that the equation is of the form

$$\phi(D_x, D_y)z = 0, \tag{6}$$

we shall say that it is a *homogeneous equation*.* If $F(x, y) \neq 0$ in (1), the equation will be called *non-homogeneous*.

Any function of x and y, which, substituted for z in a non-homogeneous equation (1), satisfies it identically, is called a *particular integral* of (1). A function satisfying the corresponding homogeneous equation (6), obtained from (1) by replacing $F(x, y)$ by zero, and containing arbitrary elements (constants or functions), will be called a *complementary function*. Because of the linear character of equation (1), the sum of a complementary function and a particular integral will evidently satisfy (1).

* Some writers define a homogeneous equation as an equation in which the dependent variable itself is absent and all the derivatives are of the same order.

The student will note the similarity between these definitions and the corresponding definitions for linear ordinary differential equations. Moreover, the important fundamental properties of linear ordinary equations and their solutions have analogues in the field of linear partial differential equations. A basic theorem, which is important to us in later work, and which may be easily proved, is the following.

THEOREM. *If $z_1, z_2, \cdots, z_n$ are n solutions of a homogeneous linear partial differential equation, then $c_1z_1 + c_2z_2 + \cdots + c_nz_n$, where the c's are any constants, is also a solution.*

62. Reducible homogeneous equations. As stated in Art. 61, the operator $\phi(D_x, D_y)$ in a linear partial differential equation is a polynomial in D_x and D_y. When the coefficients in the operator are constants, we have what is known as a linear partial differential equation with constant coefficients.

In this and the two following articles, we shall consider the problem of finding complementary functions of linear equations with constant coefficients, that is, we shall seek solutions of the homogeneous equation

$$\phi(D_x, D_y)z = 0, \tag{1}$$

in which the coefficients of the operator $\phi(D_x, D_y)$ are constants.

For convenience, we separate equations of the form (1) into two classes, which we call reducible and irreducible equations respectively. By a *reducible equation* we shall mean an equation in which the operator $\phi(D_x, D_y)$ can be resolved into factors each of which is of the first degree in D_x and D_y. An equation whose operator cannot be so factored we call *irreducible*. For example,

$$(D_x^2 - D_xD_y - D_x + D_y)z \equiv (D_x - D_y)(D_x - 1)z = 0$$

is a reducible equation, but

$$(D_x^3 - D_xD_y)z \equiv D_x(D_x^2 - D_y)z = 0$$

is irreducible since the quadratic factor $D_x^2 - D_y$ cannot be resolved into linear factors.

Consider now a reducible homogeneous equation of order m, which may be written in the form

$$\phi(D_x, D_y)z \equiv (\alpha_1D_x - \beta_1D_y - \gamma_1)\cdots(\alpha_mD_x - \beta_mD_y - \gamma_m)z = 0, \tag{2}$$

where the α's, β's and γ's are constants, some of which may be zero. Evidently any solution of the Lagrange equation

$$(\alpha_m D_x - \beta_m D_y - \gamma_m)z \equiv \alpha_m p - \beta_m q - \gamma_m z = 0 \tag{3}$$

will be a solution of (2). For, if $z = f(x, y)$ satisfies (3), so that $(\alpha_m D_x - \beta_m D_y - \gamma_m)f \equiv 0$, we have

$$\phi(D_x, D_y)f \equiv [(\alpha_1 D_x - \beta_1 D_y - \gamma_1) \cdots (\alpha_{m-1} D_x - \beta_{m-1} D_y - \gamma_{m-1})](0) \equiv 0.$$

The Lagrange equation (3) has the subsidiary equations

$$\frac{dx}{\alpha_m} = \frac{dy}{-\beta_m} = \frac{dz}{\gamma_m z}.$$

From the first two ratios, we get $\alpha_m y + \beta_m x = a$. If $\alpha_m \neq 0$, the first and third ratios give us $\log z = \gamma_m x/\alpha_m + \log b$, whence the general integral of (3) is

$$z = e^{\gamma_m x/\alpha_m} f_m(\alpha_m y + \beta_m x), \tag{4}$$

where f_m is an arbitrary function of its argument; and, if $\beta_m \neq 0$, we alternatively get as the general integral of (3),

$$z = e^{-\gamma_m y/\beta_m} f_m(\alpha_m y + \beta_m x). \tag{5}$$

Since α_m and β_m cannot both be zero if $\alpha_m D_x - \beta_m D_y - \gamma_m$ is to be a linear operative factor of $\phi(D_x, D_y)$, at least one of the forms (4) or (5) will serve in all cases. If $\alpha_m \neq 0$ and $\beta_m \neq 0$, either (4) or (5) may be used at pleasure. For definiteness, we suppose form (4) to be permissible.

Now the linear factors in (2) may be arranged in any order. Consequently each factor $\alpha_j D_x - \beta_j D_y - \gamma_j$ $(j = 1, 2, \cdots, m)$ will give rise to a solution of equation (2). By the theorem of Art. 61, it therefore follows that their sum,

$$z = e^{\gamma_1 x/\alpha_1} f_1(\alpha_1 y + \beta_1 x) + \cdots + e^{\gamma_m x/\alpha_m} f_m(\alpha_m y + \beta_m x), \tag{6}$$

where $f_1, f_2, \cdots, f_m$ are all arbitrary functions, will also satisfy equation (2). If any $\alpha_j = 0$, the corresponding term in (6) must, of course, be replaced by a term of the form (5).

If no two of the m linear factors of $\phi(D_x, D_y)$ are linearly dependent, that is, if no factor is merely some constant times another factor, it is apparent that equation (6) gives us a solution of (2) involving m essential arbitrary functions.

Example 1. Solve the third order equation

$$(D_x^2D_y + D_xD_y^2 - 2D_x^2 - 3D_xD_y + 2D_x)z = 0.$$

Solution. Here the operator has three distinct linear factors, and the equation can be written as

$$D_x(D_x + D_y - 1)(D_y - 2)z = 0.$$

Thus we have

$\alpha_1 = 1, \beta_1 = 0, \gamma_1 = 0;\quad \alpha_2 = 1, \beta_2 = -1, \gamma_2 = 1;\quad \alpha_3 = 0, \beta_3 = -1, \gamma_3 = 2.$

Hence the given equation has the solution

$$z = f_1(y) + e^xf_2(y - x) + e^{2y}f_3(x),$$

containing three arbitrary functions.

When the operator has repeated factors, the process described will lead to a solution containing fewer than m essential arbitrary functions. Let us consider a simple instance of repeated factors, namely, an equation of second order with two equal factors,

$$(\alpha D_x - \beta D_y - \gamma)^2z \equiv (\alpha D_x - \beta D_y - \gamma)(\alpha D_x - \beta D_y - \gamma)z = 0. \quad (7)$$

If we set $(\alpha D_x - \beta D_y - \gamma)z = u$, we have first of all to solve the Lagrange equation $(\alpha D_x - \beta D_y - \gamma)u = 0$. As before, supposing $\alpha \neq 0$, we find $u = e^{\gamma x/\alpha}f(\alpha y + \beta x)$, whence we have

$$(\alpha D_x - \beta D_y - \gamma)z \equiv \alpha p - \beta q - \gamma z = e^{\gamma x/\alpha}f(\alpha y + \beta x).$$

This is another Lagrange equation, having the subsidiary equations

$$\frac{dx}{\alpha} = \frac{dy}{-\beta} = \frac{dz}{\gamma z + e^{\gamma x/\alpha}f(\alpha y + \beta x)}.$$

From the first two ratios we again get $\alpha y + \beta x = a$. Using this in the first and third ratios, we find

$$\frac{dz}{dx} - \frac{\gamma}{\alpha}z = \frac{1}{\alpha}e^{\gamma x/\alpha}f(a).$$

This linear ordinary differential equation of first order has $e^{-\gamma x/\alpha}$ as integrating factor (Art. 4). Consequently we get

$$e^{-\gamma x/\alpha}z = \frac{1}{\alpha}f(a)x + b.$$

Equation (7) therefore has the solution

$$z = e^{\gamma x/\alpha}[xf_1(\alpha y + \beta x) + f_2(\alpha y + \beta x)], \quad (8)$$

where $f_1(\alpha y + \beta x) \equiv f(\alpha y + \beta x)/\alpha$ and $f_2(\alpha y + \beta x)$ are arbitrary functions. Alternatively, supposing $\beta \neq 0$,

$$z = e^{-\gamma y/\beta}\,[yf_1(\alpha y + \beta x) + f_2(\alpha y + \beta x)] \tag{9}$$

is a permissible form of solution of (7).

By extending this sort of reasoning, we may similarly show that a k-fold factor $(\alpha D_x - \beta D_y - \gamma)^k$ in the operator $\phi(D_x, D_y)$ gives rise to a portion of the complementary function which may be expressed as

$$z = e^{\gamma x/\alpha}\,[x^{k-1}f_1(\alpha y + \beta x) + x^{k-2}f_2(\alpha y + \beta x) + \cdots + f_k(\alpha y + \beta x)], \tag{10}$$

valid whenever $\alpha \neq 0$, or as

$$z = e^{-\gamma y/\beta}\,[y^{k-1}f_1(\alpha y + \beta x) + y^{k-2}f_2(\alpha y + \beta x) + \cdots + f_k(\alpha y + \beta x)], \tag{11}$$

valid for $\beta \neq 0$.

By suitably choosing one or both of these forms, we can obtain a solution, of a reducible homogeneous equation of order m, containing m essential arbitrary *functions*, whatever the multiplicity of the various linear factors in the operator.

Example 2. Solve the fifth order equation

$$(D_x^2D_y^2 + 6D_x^3D_y + 9D_x^3)z = 0.$$

Solution. Factoring the operator, we write this equation in the form

$$D_x^3(D_y + 3)^2z = 0.$$

Corresponding to the triple factor D_x^3, we have the part $x^2f_1(y) + xf_2(y) + f_3(y)$. Corresponding to the double factor $(D_y + 3)^2$, we have $e^{-3y}\,[yf_4(x) + f_5(x)]$. Hence the required solution is

$$z = x^2f_1(y) + xf_2(y) + f_3(y) + e^{-3y}\,[yf_4(x) + f_5(x)],$$

containing five arbitrary functions.

EXERCISES

Solve the following equations.

1. $(3D_x^2 - 13D_xD_y + 4D_y^2)z = 0.$
2. $(2D_x^2 + 7D_xD_y + 5D_y^2)z = 0.$
3. $(2D_x^2 - 3D_xD_y - 2D_y^2)z = 0.$
4. $(D_x^3 - D_xD_y^2)z = 0.$
5. $(3D_x^2D_y - 4D_xD_y^2)z = 0.$
6. $(D_x^2D_y^2 + 2D_xD_y^3)z = 0.$
7. $(D_x^2D_y - 4D_xD_y^2 + 4D_y^3)z = 0.$
8. $(4D_x^4 + 4D_x^3D_y + D_x^2D_y^2)z = 0.$
9. $(7D_xD_y^3 - D_y^4)z = 0.$
10. $(5D_x^3D_y^2 + 6D_x^2D_y^3)z = 0.$
11. $(D_x^3 - 6D_x^2D_y + 12D_xD_y^2 - 8D_y^3)z = 0.$
12. $(D_x^4D_y^2 + 3D_x^3D_y^3 + 3D_x^2D_y^4 + D_xD_y^5)z = 0.$

13. $(3D_x^2 + 2D_xD_y - 4D_x)z = 0.$

14. $(D_x^2 - 2D_xD_y + D_y^2 - D_x + D_y)z = 0.$

15. $(D_x^2 + 2D_xD_y + D_y^2 + 4D_x + 4D_y + 4)z = 0.$

16. $(D_x^3 + 2D_x^2D_y + D_xD_y^2 - D_x^2 - 2D_xD_y - D_y^2)z = 0.$

17. $(2D_x^3 + D_x^2D_y - 12D_x^2 - 6D_xD_y + 18D_x + 9D_y)z = 0.$

18. $(D_x^4D_y^2 - 2D_x^3D_y^2 + D_x^2D_y^2)z = 0.$

19. $(D_x^2D_y^2 - 2D_x^2D_y - 2D_xD_y^2 + D_x^2 + 4D_xD_y + D_y^2 - 2D_x - 2D_y + 1)z = 0.$

20. $(D_x^4 - 2D_x^2D_y^2 + D_y^4 + 2D_x^3 - 2D_x^2D_y - 2D_xD_y^2 + 2D_y^3 + D_x^2 - 2D_xD_y + D_y^2)z = 0.$

63. Irreducible homogeneous equations. When the operator of a homogeneous equation with constant coefficients,

$$\phi(D_x, D_y)z = 0, \tag{1}$$

cannot be resolved into linear factors, that is, when (1) is an irreducible equation, the procedure of Art. 62 does not apply. We then employ the following technique, suggested by the fact that an exponential function is reproduced, possibly multiplied by some factor, under repeated differentiation. Since

$$D_x^j D_y^k e^{ax+by} = a^j b^k e^{ax+by}$$

$(j, k = 0, 1, 2, \cdots)$, where a and b are any constants, it follows that

$$\phi(D_x, D_y)e^{ax+by} = \phi(a, b)e^{ax+by}. \tag{2}$$

Consequently the exponential function e^{ax+by} will satisfy equation (1) if and only if a and b are such that

$$\phi(a, b) = 0. \tag{3}$$

For any choice of a, relation (3), called the *auxiliary equation*, yields a certain number of distinct values of b; or, conversely, each choice of b gives us a definite number of suitable values of a.

Hence, if (a_1, b_1), (a_2, b_2), $\cdots$, (a_n, b_n) are any n value pairs satisfying (3), where n is any positive integer, we get by the above argument coupled with the theorem of Art. 61, a solution

$$z = c_1e^{a_1x+b_1y} + c_2e^{a_2x+b_2y} + \cdots + c_ne^{a_nx+b_ny}, \tag{4}$$

where c_1, c_2, $\cdots$, c_n are arbitrary constants. Although a solution of the form (4) contains no arbitrary functions, this method permits us to obtain a solution involving as many arbitrary constants as we wish.

The pairs (a_j, b_j) need not, and in some cases can not, be real number pairs. For complex values, the Euler relation

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{5}$$

where $i = \sqrt{-1}$, will be found useful. Thus, if the auxiliary equation (3) is $a^4 + b^2 = 0$, and we take for a any real number, so that $b = \pm ia^2$ is pure imaginary, a typical solution of the original differential equation is

$$e^{ax+ia^2y} + e^{ax-ia^2y} = e^{ax}(e^{ia^2y} + e^{-ia^2y})$$
$$= 2e^{ax}\cos a^2y,$$

by (5).

Evidently the method here employed is valid also for reducible equations, but for our present purpose we shall endeavor to find solutions of as general a form as possible. Hence, if an operator has some linear factors and some irreducible factors of higher degree, we find part of the solution, corresponding to the linear factors, containing arbitrary functions, and the remainder of the solution, corresponding to the irreducible factors, containing arbitrary constants.

Example 1. Solve the equation

$$(2D_x^2D_y - D_y^2)z = 0.$$

Solution. Here the operator has one linear factor, D_y, and an irreducible factor, $2D_x^2 - D_y$. Corresponding to the latter, we have the auxiliary equation

$$2a^2 - b = 0.$$

Hence our equation has the solution

$$z = f(x) + c_1e^{a_1x+2a_1^2y} + c_2e^{a_2x+2a_2^2y} + \cdots + c_ne^{a_nx+2a_n^2y},$$

where $f(x)$ is an arbitrary function of x, n is any positive integer, and the a's and c's are arbitrary constants.

In addition to the exponential function solutions (4), it is possible also to get as many other particular solutions as desired. These can be found from the exponential functions as follows. Let $b = g(a)$ denote any one expression for b in terms of a, obtained from the auxiliary equation (3). Regarding $e^{ax+by} = e^{ax+g(a)y}$ as a function of a, with x and y now playing the roles of parameters, expand this exponential function in a Maclaurin series. In this series,

$$e^{ax+g(a)y} = P_0(x, y) + P_1(x, y)a + \cdots + P_n(x, y)a^n + \cdots, \tag{6}$$

the coefficients $P_0, P_1, P_2, \cdots$ will be functions of x and y.

Series (6), representing a function which satisfies the differential equation (1) for every permissible value of a, will itself formally satisfy (1). Hence substitution of series (6) in (1) gives us a series which is identically zero for every a:

$$\phi(D_x, D_y)P_0 + \phi(D_x, D_y)P_1 a + \cdots + \phi(D_x, D_y)P_n a^n + \cdots \equiv 0.$$

Consequently every coefficient $\phi(D_x, D_y)P_n$ $(n = 0, 1, 2, \cdots)$ must be zero; that is, each of these coefficients $P_n(x, y)$ in the Maclaurin expansion of $e^{ax+g(a)y}$ is a solution of (1).

Example 2. Find polynomial solutions, up to the fourth degree, of the irreducible equation

$$(2D_x^2 - D_y)z = 0.$$

Solution. The auxiliary equation, $2a^2 - b = 0$, yields one value of b in terms of a, whence we have the exponential solution $z = e^{ax+2a^2y} = G(a)$, say. Now we find

$$\begin{aligned} G(a) &= e^{ax+2a^2y}, & G(0) &= 1,\\ G'(a) &= (x + 4ay)e^{ax+2a^2y}, & G'(0) &= x,\\ G''(a) &= [(x + 4ay)^2 + 4y]e^{ax+2a^2y}, & G''(0) &= x^2 + 4y, \end{aligned}$$

and so on, so that the Maclaurin series for $G(a)$ is

$$1 + xa + \frac{x^2 + 4y}{2!}a^2 + \frac{x^3 + 12xy}{3!}a^3 + \frac{x^4 + 24x^2y + 48y^2}{4!}a^4 + \cdots.$$

Therefore $1, x, x^2 + 4y, x^3 + 12xy, x^4 + 24x^2y + 48y^2$ are a set of polynomial solutions; of course, any constant times each of these, and linear combinations of them, are also solutions.

We next consider the possibility of finding a solution of (1) in the form of an infinite series. Instead of taking a finite number n of terms, as in (4), we now generalize to an infinite series,

$$z = c_1e^{a_1x+b_1y} + c_2e^{a_2x+b_2y} + \cdots + c_ne^{a_nx+b_ny} + \cdots, \tag{7}$$

where (a_n, b_n) $(n = 1, 2, 3, \cdots)$ are number pairs satisfying the auxiliary equation (3). The series (7) will formally satisfy (1), but in order that the symbol (7) shall have meaning, it is necessary that this series converge.

The convergence or divergence of (7) will depend upon the values assigned to x and y, the number pairs (a_n, b_n), and the coefficients c_n. In physical applications, the ranges of the variables x and y are usually given beforehand, and initial and boundary conditions serve to fix the characters of the pairs (a_n, b_n) and the c's. Thus a definite physical problem will in general lead to a unique series solution, which we can then test for convergence.

Later in this chapter we shall discuss a few physical problems whose solutions are expressible in the form of infinite series. At this point we shall consider merely a theoretical example.

Example 3. Show that the differential equation

$$(2D_x^2 - D_y)z = 0 \tag{8}$$

has the series solution

$$e^{x+2y} + e^{\sqrt{2}x+4y} + e^{\sqrt{3}x+6y} + \cdots + e^{\sqrt{n}x+2ny} + \cdots, \tag{9}$$

and find the real values of x and y for which this series converges.

Solution. The auxiliary equation, $2a^2 - b = 0$, evidently is satisfied by corresponding pairs from the sets

$$a = 1, \sqrt{2}, \sqrt{3}, \cdots, \sqrt{n}, \cdots, \qquad b = 2, 4, 6, \cdots, 2n, \cdots.$$

Hence, or by direct substitution, we see that the series (9) formerly satisfies the equation (8).

To investigate convergence, we use Cauchy's ratio test. Letting u_n denote the nth term of (9), we have

$$\frac{u_{n+1}}{u_n} = \frac{e^{\sqrt{n+1}x+2(n+1)y}}{e^{\sqrt{n}x+2ny}} = e^{(\sqrt{n+1}-\sqrt{n})x+2y}.$$

Now

$$\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = \lim_{n\to\infty} \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0,$$

and consequently, for any value of x,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = e^{2y} \lim_{n\to\infty} e^{(\sqrt{n+1}-\sqrt{n})x} = e^{2y}.$$

By Cauchy's ratio test, this limit must be less than unity for convergence. Since $e^{2y} < 1$ for $y < 0$, we conclude that series (9) converges for all real values of x and negative values of y.

64. Separation of variables. We turn now to an alternative method of finding particular solutions, which may be combined to yield series solutions, of a linear homogeneous equation with constant coefficients. This method, which is of considerable utility in connection with physical problems, often applies also to linear homogeneous equations with variable coefficients, and clearly exhibits various types of solutions obtainable in a given case. The process is best explained by means of an example; we use as an illustration the differential equation of Examples 2 and 3, Art. 63.

Example. Find solutions, each in the form of a product of a function of x alone and a function of y alone, of the differential equation

$$(2D_x^2 - D_y)z = 0. \tag{1}$$

Solution. We wish to find a solution of (1) in the form

$$z = X(x) \cdot Y(y), \tag{2}$$

where, as indicated by the functional notation, X involves x only and Y involves y only. Substituting the product (2) in (1), we get

$$2X''Y - XY' = 0, \tag{3}$$

in which the accents denote differentiation:

$$X'' = \frac{d^2X}{dx^2}, \qquad Y' = \frac{dY}{dy}.$$

At the start, we have of course no assurance that our differential equation will possess a solution of the form (2), and therefore the method is in this respect a tentative one. Success will depend upon our ability to perform the next step, which is to separate the variables in (3). Here separation of the variables is possible, and we may write (3) in the form

$$\frac{X''}{X} = \frac{Y'}{2Y}. \tag{4}$$

We have placed the constant 2 in the right member for later convenience, but it may also remain in the left member.

Now the left member of (4), involving only x, does not change when y is varied, nor does its equal, the right member, involving only y, change when x is varied. Thus neither of these equal expressions can change when x and y are both altered, and therefore the two members must be equal to a constant, say k:

$$\frac{X''}{X} = \frac{Y'}{2Y} = k. \tag{5}$$

Hence, we get, from (5), two linear ordinary differential equations,

$$\frac{d^2X}{dx^2} - kX = 0, \tag{6}$$

$$\frac{dY}{dy} - 2kY = 0. \tag{7}$$

The problem of solving the given partial differential equation (1) is consequently replaced by the problem of solving the two ordinary differential equations (6) and (7). The constant k may be given any value, but that value must be the same for both (6) and (7) in order that (4) hold.

The nature of the solutions of equations (6) and (7) will depend upon the value given to k, which may be positive, negative, or zero.* By the usual methods (Arts. 2, 11), we may find real solutions of (6) and (7) for each of these three kinds of values for k. Doing this, and inserting the results in (2), we get three types of solutions of (1):

$$z = (c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x})e^{2ky}, \qquad k > 0, \tag{8}$$

$$z = (c_3 \sin \sqrt{-k}x + c_4 \cos \sqrt{-k}x)e^{2ky}, \qquad k < 0, \tag{9}$$

$$z = c_5 x + c_6, \qquad k = 0. \tag{10}$$

In these relations, the c's are arbitrary constants and are thus at our disposal, as is k except for the stated restrictions.

Evidently two or more solutions of any or all of these three types can be linearly combined to produce more general solutions of (1). For instance, the series solution (9) in Art. 63 is obtainable by taking a series of terms all of type (8) with $c_1 = 1$, $c_2 = 0$, and $k = 1, 2, 3, \cdots$.

EXERCISES

Using the method of Example 1, Art. 63, solve each of the equations in Exercises 1–6.

1. $(D_x + 3D_y^2)z = 0.$ **2.** $(5D_x^3 - D_xD_y + D_x)z = 0.$

3. $(D_x^2D_y^2 - D_x^3 - D_y^3 + D_xD_y)z = 0.$ **4.** $(D_x^4 + 2D_x^2D_y + D_y^2)z = 0.$

5. $(D_x^2 + D_y^4)z = 0.$ **6.** $(D_x^8 + 2D_x^4D_y^2 + D_y^4)z = 0.$

7–12. By the method of Example 2, Art. 63, find particular solutions of each of the equations of Exercises 1–6. Use four terms of each Maclaurin expansion.

13. Show that the differential equation of Example 3, Art. 63, has the series solution $z = u_1 + u_2 + \cdots + u_n + \cdots$, where the nth term u_n is given by

$$\log u_n = n^{1/4}x + 2n^{1/2}y,$$

and find the values of x and y for which this series converges.

14. If, in Exercise 13, u_n is given by

$$\log u_n = nx + 2n^2y,$$

find the values of x and y for which this series converges.

15. If, in Exercise 13, $u_n = e^{-2n^2y} \cos nx$, find the values of x and y for which this series converges.

* In some problems it is desirable to give k complex values, but we shall here restrict ourselves to real values of k.

16. Show that the differential equation of Example 3, Art. 63, has the series solution

$$z = 2(e^{-2y}\sin x - \tfrac{1}{2}e^{-8y}\sin 2x + \tfrac{1}{3}e^{-18y}\sin 3x - \cdots),$$

and find the values of x and y for which this series converges. Show also that, when $y = 0$, the series represents the function x in the interval $-\pi < x < \pi$.

17. Using the method of Art. 64, show that the equation $(D_x^2 - D_y^2 - 2D_y)z = 0$ has five types of solutions, according as $k < -1, k = -1, -1 < k < 0, k = 0, k > 0$.

18. Show that the equation $(D_x^2 - D_y^2 + 4D_x - 6D_y)z = 0$ has five types of solutions, according as $k < -9, k = -9, -9 < k < -4, k = -4, k > -4$.

19. Show that the equation $(D_x^2 - D_y^2 + 2D_x - 2D_y)z = 0$ has only three types of solutions, according as $k < -1, k = -1, k > -1$. Show also that this equation has the solution $z = f_1(y + x) + e^{-2x}f_2(y - x)$, where f_1 and f_2 are arbitrary functions.

20. The method of Art. 64 may sometimes be applied to linear partial differential equations involving more than two independent variables. In the three-dimensional form of Laplace's equation (cf. Art. 32),

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0,$$

set $w = X(x)\cdot Y(y)\cdot Z(z)$, and separate the variables to get in turn the relations

$$\frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} = k_1,$$

$$\frac{Y''}{Y} = -\frac{Z''}{Z} - k_1 = k_2,$$

where k_1 and k_2 are constants. Hence show that Laplace's equation has thirteen types of solutions of the assumed form.

65. Non-homogeneous equations. We consider next the problem of solving a non-homogeneous linear equation with constant coefficients,

$$\phi(D_x, D_y)z = F(x, y). \tag{1}$$

Now a solution of an equation of type (1) is made up of the sum of a complementary function and a particular integral (Art. 61). By the methods of Arts. 62–64 we can find a complementary function; we have therefore to find a particular integral of (1), that is, any function of x and y such that when it is operated upon with $\phi(D_x, D_y)$, the right member $F(x, y)$ is produced.

The designations reducible and irreducible may conveniently be applied to equation (1) also. Thus, the non-homogeneous equation (1) will be called reducible or irreducible according as the corresponding homogeneous equation, obtained by replacing $F(x, y)$ by zero, is reducible or irreducible.

When an equation (1), of order m, is reducible, a particular integral may be found by treating successively m Lagrange equations. For, let the operator $\phi(D_x, D_y)$ be resolved into linear factors (some of which may be repeated), and write (1) as

$$\phi(D_x, D_y)z \equiv (\alpha_1 D_x - \beta_1 D_y - \gamma_1) \cdots (\alpha_m D_x - \beta_m D_y - \gamma_m)z = F(x, y). \quad (2)$$

Setting

$$(\alpha_2 D_x - \beta_2 D_y - \gamma_2) \cdots (\alpha_m D_x - \beta_m D_y - \gamma_m)z = u, \quad (3)$$

we have first to solve the Lagrange equation

$$(\alpha_1 D_x - \beta_1 D_y - \gamma_1)u = F(x, y). \quad (4)$$

Having found a solution $u = u(x, y)$ of (4), we insert this expression in the linear equation (3) of order $m - 1$. The process may then be repeated for the new equation of lower order, and so on through m stages.

We shall suppose that a complementary function of (1) is known, so that only a particular integral need be found. Accordingly, in solving each Lagrange equation of the form (4), we may omit arbitrary functions. Our final solution of (1) should, of course, consist of the sum of the complementary function and the particular integral we have found.

Example 1. Solve the equation

$$(D_x^2 - 3D_xD_y + 2D_y^2 - D_x + 2D_y)z = (2 + 4x)e^{-y}.$$

Solution. This is a reducible equation, for we may write it as

$$(D_x - 2D_y)(D_x - D_y - 1)z = (2 + 4x)e^{-y}.$$

We therefore have, as one form of complementary function, the expression $f_1(y + 2x) + e^{-y}f_2(y + x)$, where f_1 and f_2 are arbitrary.

To find a particular integral, let $u = (D_x - D_y - 1)z$, so that

$$(D_x - 2D_y)u = (2 + 4x)e^{-y},$$

for which the subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-2} = \frac{du}{(2 + 4x)e^{-y}}.$$

We then get $y + 2x = a$, and, using this relation,

$$du = (-1 - 2x)e^{-y}\,dy = (y - a - 1)e^{-y}\,dy,$$

$$u = (a - y)e^{-y} + a' = 2xe^{-y} + a'.$$

The general integral of this first Lagrange equation is $u = 2xe^{-y} + f_1(y + 2x)$, but since we know that an arbitrary function of $y + 2x$ appears in the complementary function of the given equation, we need not include it here. Consequently we next solve the equation

$$(D_x - D_y - 1)z = 2xe^{-y},$$

which has the subsidiary equations

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{z + 2xe^{-y}}.$$

From these we get $y + x = b$ and

$$\frac{dz}{dy} + z = -2xe^{-y} = -2(b - y)e^{-y}.$$

This linear equation has e^y as integrating factor, whence we find

$$ze^y = -2by + y^2 + b' = -y^2 - 2xy + b'.$$

Again, we omit the arbitrary function, $f_2(y + x)$. Hence a particular integral of the original equation is $-(y^2 + 2xy)e^{-y}$, and our final answer may be expressed as

$$z = f_1(y + 2x) + e^{-y}f_2(y + x) - (y^2 + 2xy)e^{-y}.$$

Variations in the process, such as changes in the order of the factors of the operator, will sometimes lead to different particular integrals. These differences can be accounted for by making suitable changes in the arbitrary functions of the complementary function. For instance, x^2e^{-y} is also a particular integral of the above differential equation, as may readily be verified. Now the solution found for Example 1 can be changed as follows. We have

$$\begin{aligned} z &= f_1(y + 2x) + e^{-y}[f_2(y + x) - (y^2 + 2xy)] \\ &= f_1(y + 2x) + e^{-y}[f_2(y + x) - (y + x)^2 + x^2] \\ &= f_1(y + 2x) + e^{-y}g(y + x) + x^2e^{-y}, \end{aligned}$$

where $g(y + x)$ is a new arbitrary function equivalent to $f_2(y + x) - (y + x)^2$. The latter form of solution exhibits x^2e^{-y} as a particular integral.

When the given differential equation (1) is irreducible, reduction to Lagrange equations cannot be effected. Choice of a method for finding a particular integral of an irreducible equation will, in general, depend upon the form of the right member $F(x, y)$. We shall develop here a few processes for treating right members of particular but common forms.

A solution of equation (1) may be represented symbolically as

$$z = \frac{1}{\phi(D_x, D_y)} \cdot F(x, y), \tag{5}$$

in which $1/\phi(D_x, D_y)$ is to be regarded as an inverse operator acting on F. Now the possession by an operator of algebraic properties, such as factorability, suggests that we perform the indicated division of unity by $\phi(D_x, D_y)$, at least through a certain number of stages, and then, if possible, operate on F with the result.

Such division can usually be performed in two ways. For instance, we have

$$\frac{1}{2D_x^2 - D_y} = \frac{1}{2D_x^2}\frac{1}{1 - (D_y/2D_x^2)}$$

$$= \frac{1}{2D_x^2}\left[1 + \frac{D_y}{2D_x^2} + \frac{D_y^2}{4D_x^4} + \frac{D_y^3}{8D_x^6(1 - D_y/2D_x^2)}\right] \tag{6}$$

and

$$\frac{1}{2D_x^2 - D_y} = -\frac{1}{D_y}\frac{1}{1 - (2D_x^2/D_y)}$$

$$= -\frac{1}{D_y}\left[1 + \frac{2D_x^2}{D_y} + \frac{4D_x^4}{D_y^2} + \frac{8D_x^6}{D_y^3(1 - 2D_x^2/D_y)}\right]. \tag{7}$$

Since a positive integral power of D_x, or of D_y, denotes differentiation, we may naturally interpret $1/D_x$ and $1/D_y$ as integrating operators. Moreover, being concerned here with finding as simple a particular integral as possible, we may agree to omit an arbitrary element of integration.

Now when $F(x, y)$ is a polynomial in x and y, a finite number of partial differentiations, with respect to x or with respect to y, produces zero. Hence, if we first perform the differentiation processes in an expansion such as (6) or (7), we need not evolve interpretations of the final terms in these expansions when operating on a polynomial. We shall not attempt to justify or generalize the division process beyond this point, but shall merely employ the method tentatively; results thus obtained can readily be checked.

Example 2. Find a particular integral of the equation

$$(2D_x^2 - D_y)z = 3x - y^2 + 2x^2y.$$

$(2D_x^2 - D_y)z = 3x - y^2 + 2x^2y$

Solution. Since the right member is of the second degree in either x or y, we use expansion (7), of which only two terms are needed, rather than (6), in which the third term produces a non-zero result. Then we have

$$\begin{aligned} z &= -\frac{1}{D_y}\left(1 + \frac{2}{D_y} D_x^2\right)\cdot(3x - y^2 + 2x^2y) \\ &= -\frac{1}{D_y}\left[3x - y^2 + 2x^2y + \frac{2}{D_y}(4y)\right] \\ &= -\frac{1}{D_y}(3x - y^2 + 2x^2y + 4y^2) \\ &= -\frac{1}{D_y}(3x + 3y^2 + 2x^2y) \\ &= -3xy - y^3 - x^2y^2. \end{aligned}$$

It is easily verified that this expression satisfies the given equation, and therefore serves as a particular integral.

We next develop the so-called shifting formula (cf. Art. 10), useful in many connections,

$$\phi(D_x, D_y)(we^{ax+by}) = e^{ax+by}\phi(D_x + a, D_y + b)w, \tag{8}$$

where a and b are any constants and w is any function of x and y. We have

$$\begin{aligned} D_x(we^{ax+by}) &= e^{ax+by}D_xw + awe^{ax+by} \\ &= e^{ax+by}(D_x + a)w, \\ D_x^2(we^{ax+by}) &= D_x[e^{ax+by}(D_x + a)w] \\ &= e^{ax+by}(D_x^2 + aD_x)w + e^{ax+by}(aD_x + a^2)w \\ &= e^{ax+by}(D_x + a)^2w, \end{aligned}$$

and, by induction, it is easy to prove that, for j any positive integer,

$$D_x^j(we^{ax+by}) = e^{ax+by}(D_x + a)^jw.$$

Likewise, operating on the latter relation k times with D_y, we find

$$D_y^kD_x^j(we^{ax+by}) = e^{ax+by}(D_y + b)^k(D_x + a)^jw,$$

whence formula (8) immediately follows.

Suppose now that the right member $F(x, y)$, or any term of it, is of the form ce^{ax+by}, where a, b, and c are constants. Then since

$$\phi(D_x, D_y)e^{ax+by} = \phi(a, b)e^{ax+by},$$

we have, if $\phi(a, b) \neq 0$, the interpretation

$$\frac{1}{\phi(D_x, D_y)} \cdot ce^{ax+by} = \frac{c}{\phi(a, b)} e^{ax+by}. \tag{9}$$

This yields the corresponding particular integral when $\phi(a, b) \neq 0$.

If $\phi(a, b) = 0$, we may use the shifting formula (8), as follows. In the differential equation,

$$\phi(D_x, D_y)z = ce^{ax+by}, \tag{10}$$

substitute $z = we^{ax+by}$, whence we get

$$\phi(D_x, D_y)(we^{ax+by}) = e^{ax+by}\phi(D_x + a, D_y + b)w = ce^{ax+by}.$$

We then have merely to find a particular integral of the equation

$$\phi(D_x + a, D_y + b)w = c. \tag{11}$$

This may often be found by inspection, otherwise by the method of Example 2. The product of e^{ax+by} and a particular integral of (11) then gives us a particular integral of (10).

Evidently terms in $F(x, y)$ of the form $cx^jy^ke^{ax+by}$ can be similarly treated. For, under the substitution $z = we^{ax+by}$, the differential equation

$$\phi(D_x, D_y)z = cx^jy^ke^{ax+by} \tag{12}$$

becomes, by (8),

$$\phi(D_x + a, D_y + b)w = cx^jy^k. \tag{13}$$

Example 3. Find a particular integral of the equation

$$(2D_x^2 - D_y)z = 4e^{x+2y} + 10e^{x-3y}.$$

Solution. Using relation (9), a particular integral corresponding to the second term on the right is found to be

$$\frac{10}{2 \cdot 1^2 - (-3)} e^{x-3y} = 2e^{x-3y}.$$

We need now find only a particular integral of

$$(2D_x^2 - D_y)z = 4e^{x+2y}.$$

Since $2 \cdot 1^2 - 2 = 0$, we set $z = we^{x+2y}$, and get by (8),

$$e^{x+2y}[2(D_x + 1)^2 - (D_y + 2)]w = 4e^{x+2y},$$

$$(2D_x^2 + 4D_x - D_y)w = 4.$$

It is easy to see that either x or $-4y$ will serve as a particular integral of this equation. Choosing the former, we then have

$$z = xe^{x+2y} + 2e^{x-3y}$$

as a particular integral of the given equation.

If a term of $F(x, y)$ is of the form $c \sin(ax + by)$, or $c \cos(ax + by)$, it is usually simplest to use the method of undetermined coefficients (cf. Art. 12), as in the following example.

Example 4. Find a particular integral of the equation

$$(2D_x^2 - D_y)z = 8 \sin(x + 2y).$$

Solution. Set $z = A \sin(x + 2y) + B \cos(x + 2y)$, where A and B are constants to be determined. Substituting in the given equation, we get

$$-2A \sin(x + 2y) - 2B \cos(x + 2y) - 2A \cos(x + 2y) + 2B \sin(x + 2y) = 8 \sin(x + 2y).$$

Then we must have $-2A + 2B = 8$, $-2B - 2A = 0$, whence $A = -2$ and $B = 2$. Therefore

$$z = -2 \sin(x + 2y) + 2 \cos(x + 2y)$$

is a particular integral.

Use of the shifting formula (8), followed by the procedure of Example 4, will serve to find a particular integral corresponding to a term of either of the forms $ce^{ax+by} \sin(a_1x + b_1y)$, $ce^{ax+by} \cos(a_1x + b_1y)$.

Evidently all the processes of Examples 2–4 may be applied also to reducible equations. For reducible equations of order higher than the second, and for some reducible equations of second order, our special methods, when applicable, will often be shorter than the reduction method of Example 1.

EXERCISES

Find a particular integral of each of the following equations.

1. $(D_x^2 - D_y^2)z = 6.$

2. $(D_x^2 - 4D_y^2)z = 3x - 4y.$

3. $(D_x^2 + 2D_y^2)z = 30xy^2.$

4. $(2D_x^2 - 3D_y^2 + D_x)z = 14e^{2x-y}.$

5. $(2D_x^2 - 3D_y^2 + D_x)z = 10e^{x-y}.$

6. $(D_x^2 - D_xD_y)z = 2xe^{x+y}.$

7. $(D_y^2 + 3D_xD_y)z = 2x + 6y - 9.$

8. $(D_x^2 - 2D_xD_y)z = 8y^2 - 8xy.$

9. $(2D_x^2 - D_xD_y + D_y)z = 13 - x.$

10. $(D_x^3 - D_y^2)z = 6y^2 - 2x^3.$

11. $(D_x^2D_y + 2D_y^2 - D_x)z = 24e^{y-2x}.$

12. $(D_x^2D_y - 2D_y^2 + D_x)z = 24e^{y-2x}.$

13. $(2D_xD_y + 3D_y^2 - D_y)z = 6 \cos(2x - 3y) - 30 \sin(2x - 3y).$

14. $(3D_x^2 - 2D_y^2 + D_x - 1)z = 3e^{x-y} \cos(x - y).$

15. $(D_xD_y^2 + D_x^2 - 2)z = 16e^{2x} \cos 3y.$

16. $(D_x^2 + D_xD_y - 2D_y^2)z = 8 \sec^2(2x + y) \tan(2x + y).$

17. $(D_x^2 + D_xD_y - D_x - D_y)z = 1/y^2.$

18. $(D_x^2 - D_y^2 - D_x + D_y)z = (x + 1)/x^2.$

19. $(D_x^2 - D_y^2 + 4D_y - 4)z = 4x(1 - y)$.
20. $(D_x^2D_y - D_xD_y^2 + D_xD_y)z = 1$.

66. Analogues of Euler equations. In this and the following three articles we shall consider linear partial differential equations with variable coefficients.

Only certain types of such equations, solvable by elementary methods, can be discussed here. We begin with a type that can be transformed into a linear equation with constant coefficients, and which is an analogue of the Euler equation (Art. 13). This type is characterized by the fact that the operator $\phi(D_x, D_y)$ consists of a sum of terms of the form

$$c_{jk}x^jy^kD_x^jD_y^k, \tag{1}$$

$(j, k = 0, 1, 2, \cdots, m)$, the c's being constants. For example, the second order type form is

$$(c_{20}x^2D_x^2 + c_{11}xyD_xD_y + c_{02}y^2D_y^2 + c_{10}xD_x + c_{01}yD_y + c_{00})z = F(x, y).$$

Changing the independent variables from x and y to u and v by means of the substitutions

$$x = e^u, \qquad y = e^v \tag{2}$$

will transform the given equation into a linear equation with constant coefficients. For, we have

$$D_xz = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{du}{dx} = \frac{1}{x}D_uz, \qquad xD_xz = D_uz,$$

$$D_yz = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial v}\frac{dv}{dy} = \frac{1}{y}D_vz, \qquad yD_yz = D_vz,$$

$$D_x^2z = \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial z}{\partial u}\right) = \frac{1}{x^2}\frac{\partial^2 z}{\partial u^2} - \frac{1}{x^2}\frac{\partial z}{\partial u}, \qquad x^2D_x^2z = D_u(D_u - 1)z,$$

$$D_xD_yz = \frac{\partial}{\partial x}\left(\frac{1}{y}D_vz\right) = \frac{1}{y}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) = \frac{1}{xy}\frac{\partial^2 z}{\partial u\partial v}, \quad xyD_xD_yz = D_uD_vz,$$

$$D_y^2z = \frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial z}{\partial v}\right) = \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2} - \frac{1}{y^2}\frac{\partial z}{\partial v}, \qquad y^2D_y^2z = D_v(D_v - 1)z,$$

and, as may be proved by induction,

$$x^jy^kD_x^jD_y^kz = D_u(D_u - 1)\cdots(D_u - j + 1)D_v(D_v - 1)\cdots(D_v - k + 1)z. \tag{3}$$

Hence each term of the form (1) becomes an expression involving powers of D_u and D_v and only constant coefficients. In non-homogeneous equations, the right member, $F(x, y)$, is of course also subjected to the transformation (2), becoming $F(e^u, e^v) = G(u, v)$, say.

After transformation, the resulting equation may then be attacked by the methods of Arts. 62–65. When z has been found as a function of u and v, the inverse transformation,

$$u = \log x, \qquad v = \log y, \tag{4}$$

serves to give us a solution of the original equation.

Example. Solve the equation

$$(x^2D_x^2 - 3xyD_xD_y + 2y^2D_y^2 + xD_x + 2yD_y)z = 6xy^2.$$

Solution. Using relations (2) and (3), we get as the transformed equation,

$$[D_u(D_u - 1) - 3D_uD_v + 2D_v(D_v - 1) + D_u + 2D_v]z = 6e^u \cdot e^{2v},$$

or

$$(D_u^2 - 3D_uD_v + 2D_v^2)z = 6e^{u+2v}.$$

Since the new operator has the factors $D_u - D_v$ and $D_u - 2D_v$, $g_1(v + u) + g_2(v + 2u)$ is a suitable complementary function. Also, by means of formula (9), Art. 65, we easily find that $z = 2e^{u+2v}$ is a particular integral. Hence the new equation has the solution

$$z = g_1(v + u) + g_2(v + 2u) + 2e^{u+2v}.$$

Returning to the original independent variables, x and y, we then get, as a solution of the given equation,

$$z = g_1(\log xy) + g_2(\log x^2y) + 2xy^2,$$

or

$$z = f_1(xy) + f_2(x^2y) + 2xy^2,$$

where f_1 and f_2 are arbitrary functions.

EXERCISES

Solve each of the following equations.

1. $(x^2D_x^2 - y^2D_y^2 + xD_x - yD_y)z = 0.$

2. $(x^2D_x^2 - 2xyD_xD_y + y^2D_y^2 + xD_x + yD_y)z = 0.$

3. $(x^2D_x^2 - xyD_xD_y - xD_x)z = 0.$

4. $(x^2D_x^2 - y^2D_y^2 - 2xD_x + 2yD_y)z = 0.$

5. $(xyD_xD_y - y^2D_y^2 - 2xD_x + 2yD_y - 2)z = 0.$

6. $(2x^2D_x^2 - 5xyD_xD_y + 2y^2D_y^2 + 2xD_x + 2yD_y)z = 5xy^3.$

7. $(2x^2D_x^2 + 7xyD_xD_y + 3y^2D_y^2 + 2xD_x + 3yD_y)z = 10y/x^3.$

8. $(3x^2D_x^2 - 8xyD_xD_y - 3y^2D_y^2 + 3xD_x - 3yD_y)z = 16.$
9. $(x^2D_x^2 - xyD_xD_y - 2y^2D_y^2 + xD_x - 2yD_y)z = 2\log(y/x).$
10. $(x^2D_x^2 - 3xyD_xD_y + 2y^2D_y^2 + xD_x + 2yD_y)z = 4x^2y\log x.$
11. $(x^2D_x^2 + 3xyD_xD_y - 4y^2D_y^2 + xD_x - 4yD_y)z = 12\sin\log y^2.$
12. $(6x^2D_x^2 - xyD_xD_y - y^2D_y^2 + 6xD_x - yD_y)z = 14xy\cos\log(x/y).$
13. $(xD_x^2D_y - yD_xD_y^2)z = 0.$
14. $(x^3D_x^3 - xy^2D_xD_y^2 + 3x^2D_x^2 - xyD_xD_y + xD_x)z = 4.$
15. $(x^2yD_x^2D_y - xy^2D_xD_y^2 - x^2D_x^2 + y^2D_y^2)z = 6x^2/y.$

67. Special types. We next consider a few special types of second order linear partial differential equations with variable coefficients. These particular forms are included in the general linear equation of second order,

$$Rr + Ss + Tt + Pp + Qq + Zz = F, \tag{1}$$

where $R, \cdots, Z, F$ involve the independent variables x and y only.

Among the simplest types are those in which five of the six coefficients in the left member of (1) are zero. In order that we shall actually have an equation of second order to deal with, R or S or T must be the non-zero coefficient, and we thus have the three possible forms

$$r \equiv \frac{\partial^2 z}{\partial x^2} = G(x, y), \tag{2}$$

$$s \equiv \frac{\partial^2 z}{\partial x \partial y} = G(x, y), \tag{3}$$

$$t \equiv \frac{\partial^2 z}{\partial y^2} = G(x, y). \tag{4}$$

We shall discuss only equation (2); the remaining two equations may be similarly treated.

Since $r \equiv \partial^2 z/\partial x^2 \equiv \partial p/\partial x$, partial integration of (2) with respect to x, that is, integration in which y is regarded as constant, will yield p. In this integration, the arbitrary element introduced may then properly be taken as an arbitrary function of y. A second integration with respect to x, in which a second arbitrary function of y appears, will then yield z.

Example 1. Find a solution, containing two arbitrary functions, of the equation

$$r = 6xy^2 - 2y.$$

Solution. One integration with respect to x gives us

$$p = 3x^2y^2 - 2xy + f(y),$$

where $f(y)$ is arbitrary. Integrating again with respect to x, we get

$$z = x^3y^2 - x^2y + xf(y) + g(y),$$

in which $g(y)$ is also arbitrary. This is the solution sought.

Of course, equations (2)–(4) are reducible linear equations with constant coefficients, and may therefore be solved by the methods of Arts. 62 and 65. However, it is usually easier directly to perform the two integrations, as in Example 1.

Next in order of simplicity are the equations

$$Rr + Pp = F, \tag{5}$$

$$Ss + Qq = F, \tag{6}$$

$$Ss + Pp = F, \tag{7}$$

$$Tt + Qq = F. \tag{8}$$

The first of these can be written as

$$R\frac{\partial p}{\partial x} + Pp = F,$$

which, with y regarded as constant, is essentially a linear ordinary differential equation of the first order, p being the dependent variable. By the usual method (Art. 4), p can be found, and integration with respect to x will then give us z.

Similarly, (6) may be expressed as

$$S\frac{\partial q}{\partial x} + Qq = F,$$

and two integrations, with respect to x and y in turn, serve to solve the problem. Equations (7) and (8), equivalently written as

$$S\frac{\partial p}{\partial y} + Pp = F, \qquad T\frac{\partial q}{\partial y} + Qq = F,$$

require first an integration with respect to y, and then a second integration, with respect to x for the former, with respect to y for the latter.

Example 2. Find a solution, containing two arbitrary functions, of the equation

$$xs + q = 4e^{2x-y}.$$

Solution. This is an equation of the form (6), and we therefore write it as

$$x\frac{\partial q}{\partial x} + q = 4e^{2x-y}.$$

This may be integrated as a linear ordinary equation, but it is easy to integrate by combination (Art. 3). For, we notice that the left member is the partial derivative with respect to x of the product xq. Hence we have

$$xq = 2e^{2x-y} + f(y),$$

$$q \equiv \frac{\partial z}{\partial y} = \frac{2}{x}e^{2x-y} + \frac{1}{x}f(y),$$

where $f(y)$ is arbitrary. Integration with respect to y now gives us the solution

$$z = -\frac{2}{x}e^{2x-y} + \frac{1}{x}f_1(y) + g(x),$$

where $f_1(y)$ is a new arbitrary function of y (such that $df_1/dy = f$) and $g(x)$ is also arbitrary.

Consider now the forms

$$Rr + Pp + Zz = F, \tag{9}$$

$$Tt + Qq + Zz = F. \tag{10}$$

Equation (9), when written as

$$R\frac{\partial^2 z}{\partial x^2} + P\frac{\partial z}{\partial x} + Zz = F,$$

is seen to be essentially a second order linear ordinary differential equation with x as independent variable and y playing the role of parameter. If, in particular, the coefficients R, P, and Z contain y but not x, this equation can be solved as a linear ordinary equation with constant coefficients (Arts. 11, 12). The arbitrary elements appearing in the solution should, of course, be arbitrary functions of y.

Similarly, form (10),

$$T\frac{\partial^2 z}{\partial y^2} + Q\frac{\partial z}{\partial y} + Zz = F,$$

may sometimes be solved by elementary methods, treating x as a parameter. Equation (10) can always be solved by elementary means if T, Q, and Z depend only upon x.

Example 3. Find a solution, containing two arbitrary functions, of the equation

$$r - 3yp + 2y^2z = 2(y - 2)e^{2x-y}.$$

Solution. This equation, which for emphasis may be expressed in operational form,

$$(D_x^2 - 3yD_x + 2y^2)z = 2(y - 2)e^{2x-y},$$

is readily solvable by the methods of Arts. 11–12. The auxiliary equation, $m^2 - 3ym + 2y^2 = 0$, has the roots y and $2y$, and consequently the complementary function is

$$e^{yx}f(y) + e^{2yx}g(y),$$

where f and g are arbitrary functions. By the method of undetermined coefficients (Art. 12), or otherwise, we find the particular integral $z = e^{2x-y}/(y - 1)$. Hence the given equation has the solution

$$z = e^{xy}f(y) + e^{2xy}g(y) + \frac{e^{2x-y}}{y-1}.$$

Finally, we consider here the forms

$$Rr + Ss + Pp = F, \tag{11}$$

$$Ss + Tt + Qq = F. \tag{12}$$

Equation (11) may be written as

$$R\frac{\partial p}{\partial x} + S\frac{\partial p}{\partial y} = F - Pp,$$

which is a linear partial differential equation of the first order with x and y as independent variables and p as dependent variable. Hence it can be attacked by the Lagrange method of Chapter IV. Since p occurs only linearly in this Lagrange equation, it is usually possible to obtain p explicitly in terms of x and y, one arbitrary function being involved. If we can integrate the result with respect to x, we shall get a solution of (11), into which a second arbitrary function, of y, is introduced.

Equation (12), written as

$$S\frac{\partial q}{\partial x} + T\frac{\partial q}{\partial y} = F - Qq,$$

can evidently be attacked in a like manner.

Example 4. Find a solution, containing two arbitrary functions, of the equation

$$xr - ys + p = y^2.$$

Solution. As this equation falls under form (11), we express it as

$$x\frac{\partial p}{\partial x} - y\frac{\partial p}{\partial y} = y^2 - p,$$

the subsidiary equations for which are

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dp}{y^2 - p}.$$

From the first two ratios we get $xy = a$. From the second and third ratios, we find

$$y\,dp - p\,dy + y^2\,dy = 0,$$

$$\frac{y\,dp - p\,dy}{y^2} + dy = 0,$$

$$\frac{p}{y} + y = b.$$

Therefore the general solution of the Lagrange equation is

$$\frac{p}{y} + y = f(xy),$$

where f is arbitrary. Then

$$p \equiv \frac{\partial z}{\partial x} = -y^2 + yf(xy),$$

and

$$z = -xy^2 + f_1(xy) + g(y),$$

where $f_1(xy)$ is a new arbitrary function such that $\partial f_1(xy)/\partial x \equiv yf_1'(xy) = yf(xy)$, and $g(y)$ is also arbitrary.

Sometimes an equation of type (11) or type (12) can also be solved by a direct first integration. For instance, the equation of Example 4 can at once be integrated with respect to x, by combination, giving us

$$xp - yq = xy^2 + \phi(y),$$

where $\phi(y)$ is arbitrary. This Lagrange equation, with z as dependent variable, can be solved to yield the same solution found above; it is left to the student to verify this.

EXERCISES

Find a solution, containing two arbitrary functions, of each of the following equations.

1. $r = 2 \sin (x - 2y)$.
2. $s = 2x - 3y^2$.
3. $t = xe^{-y}$.
4. $yr = \sec^2 x$.
5. $x^2ys = 2 - 6xy^2$.
6. $(x + y)t = 2xy$.
7. $xr + 2p = 0$.
8. $ys + yq = 1 + ye^{-x}$.
9. $ys - p = 3y^2$.
10. $yt - q = 2x$.
11. $t - q - (x + x^2)z = 2x^2(1 + xy + x^2y)$.
12. $s \cos x + q \sin x = \sin x \cos y$.
13. $ys + 2p = 3y \cos x$.
14. $t + q = x^2 \cos xy + x \sin xy$.
15. $r - yp = 2 - 2xy$.
16. $r + s = 4xy + 2y^2$.
17. $xs - 2x^2t = 1$.
18. $xyr + x^2s - yp = (x^2 - y)e^y$.
19. $x^2r + x(2 - y)p - yz = 3x^2(2 - y)$.
20. $xs + yt + q = 12xy^2$.

68. Laplace's transformation. A second order linear equation with variable coefficients,

$$Rr + Ss + Tt + Pp + Qq + Zz = F, \tag{1}$$

which does not fall under any of the types discussed in Art. 67, can sometimes be transformed into an equation solvable by elementary methods. We consider here Laplace's method of transforming equation (1) and possible ways of solving the resulting equation.

We first change the independent variables from x and y to a new pair u and v. For the present, the functional relations connecting the original and new variables will not be specified; each of the new variables u and v will be some function of x and y, to be determined when and as desired. For brevity, we use the following notation with reference to the variables u and v, analogous to our usual notation when x and y are the independent variables:

$$p' \equiv \frac{\partial z}{\partial u}, \quad q' \equiv \frac{\partial z}{\partial v}, \quad r' \equiv \frac{\partial^2 z}{\partial u^2}, \quad s' \equiv \frac{\partial^2 z}{\partial u \partial v}, \quad t' \equiv \frac{\partial^2 z}{\partial v^2}. \tag{2}$$

Remembering that z and its partial derivatives, with respect to x and y or with respect to u and v, as well as u and v themselves, are functions of x and y, we get (Art. 18)

$$p \equiv \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = p'u_x + q'v_x, \qquad q = p'u_y + q'v_y,$$

$$\begin{aligned} r \equiv \frac{\partial p}{\partial x} &= p'u_{xx} + u_x(r'u_x + s'v_x) + q'v_{xx} + v_x(s'u_x + t'v_x) \\ &= r'u_x^2 + 2s'u_xv_x + t'v_x^2 + p'u_{xx} + q'v_{xx}, \end{aligned} \tag{3}$$

$$s \equiv \frac{\partial p}{\partial y} \equiv \frac{\partial q}{\partial x} = r' u_x u_y + s'(u_x v_y + u_y v_x) + t' v_x v_y + p' u_{xy} + q' v_{xy},$$

$$t \equiv \frac{\partial q}{\partial y} = r' u_y^2 + 2s' u_y v_y + t' v_y^2 + p' u_{yy} + q' v_{yy}.$$

Substituting in (1) and rearranging, we find as the general form of the transformed equation,

$$R'r' + S's' + T't' + P'p' + Q'q' + Zz = F, \tag{4}$$

where

$$\begin{aligned} R' &= Ru_x^2 + Su_x u_y + Tu_y^2, \\ S' &= 2Ru_x v_x + S(u_x v_y + u_y v_x) + 2Tu_y v_y, \\ T' &= Rv_x^2 + Sv_x v_y + Tv_y^2, \\ P' &= Ru_{xx} + Su_{xy} + Tu_{yy} + Pu_x + Qu_y, \\ Q' &= Rv_{xx} + Sv_{xy} + Tv_{yy} + Pv_x + Qv_y. \end{aligned} \tag{5}$$

Now we have the two quantities u and v, as functions of x and y, at our disposal. Examination of the forms (5) of the coefficients in the new equation (4) shows that u and v can always be chosen so as to make vanish two of the three coefficients R', S', T', of the second derivatives in (4). For, consider the quadratic equation in θ,

$$R\theta^2 + S\theta + T = 0, \tag{6}$$

suggested by the form of R' and T' in relations (5). Suppose first that the roots of (6), which of course depend upon x and y since R, S, and T do, are distinct, and denote them by α and β. We then determine u and v as solutions, as simple as possible, of the Lagrange equations

$$\frac{\partial u}{\partial x} = \alpha \frac{\partial u}{\partial y}, \tag{7}$$

$$\frac{\partial v}{\partial x} = \beta \frac{\partial v}{\partial y}. \tag{8}$$

Consequently, since α and β satisfy (6),

$$R' = R\alpha^2 u_y^2 + S\alpha u_y^2 + Tu_y^2 = (R\alpha^2 + S\alpha + T)u_y^2 = 0,$$

$$T' = (R\beta^2 + S\beta + T)v_y^2 = 0.$$

Moreover, we have from algebra, $\alpha\beta = T/R$, $\alpha + \beta = -S/R$, and therefore

$$S' = [2R\alpha\beta + S(\alpha + \beta) + 2T]u_y v_y$$
$$= \left(2T - \frac{S^2}{R} + 2T\right)u_y v_y$$
$$= -\frac{S^2 - 4RT}{R} u_y v_y.$$

Since, by supposition, $\alpha \neq \beta$, the discriminant $S^2 - 4RT$ of the quadratic (6) is not zero, and consequently $S' \neq 0$.

With u and v determined by equations (7) and (8), we thus have $R' = T' = 0$, $S' \neq 0$, and the transformed equation (4) may be written in the form

$$s' + Lp' + Mq' + Nz = G,$$

or

$$\frac{\partial^2 z}{\partial u \partial v} + L\frac{\partial z}{\partial u} + M\frac{\partial z}{\partial v} + Nz = G. \tag{9}$$

In several cases, equation (9) can be solved. If it happens that the coefficients L and N both vanish, we have an instance of form (6) of Art. 67; or, if $M = N = 0$, form (7) of Art. 67 arises.

Two further possibilities may be considered. On the one hand, we may write (9) as

$$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v} + Lz\right) + M\left(\frac{\partial z}{\partial v} + Lz\right) + \left(N - \frac{\partial L}{\partial u} - LM\right)z = G. \tag{10}$$

Then if $N - L_u - LM = 0$, (10) becomes the first order equation

$$\frac{\partial w}{\partial u} + Mw = G, \tag{11}$$

where

$$w = \frac{\partial z}{\partial v} + Lz. \tag{12}$$

Solving these two Lagrange equations in turn, we get z in terms of u and v, and thence z as a function of x and y. On the other hand, (9) may be expressed as

$$\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + Mz\right) + L\left(\frac{\partial z}{\partial u} + Mz\right) + \left(N - \frac{\partial M}{\partial v} - LM\right)z = G,$$

so that, if $N - M_v - LM = 0$, we have merely to solve

$$\frac{\partial w}{\partial v} + Lw = G, \tag{13}$$

and then

$$w = \frac{\partial z}{\partial u} + Mz. \tag{14}$$

When none of these conditions is fulfilled, it is possible to make further transformations, one of which may lead to a tractable form, but we shall not go into these matters here (see Exercises 11 and 12, following Art. 69). In order to keep the argument from becoming too involved, we have also passed over a few exceptional cases; for instance, we have tacitly assumed that $R \neq 0$, thereby making (6) truly a quadratic equation. This and certain other points will be left to the exercises.

We have still to consider the case in which the quadratic equation (6) has equal roots, so that

$$S^2 - 4RT = 0, \tag{15}$$

and (6) yields only one function, say α. We then determine u from (7), and take $v = y$. As a consequence, $R' = 0$ as before, and $T' = T$, from relations (5). If, in the original equation (1), $T = 0$, (6) has the root zero, and, since we are now supposing that (6) has two equal roots, this implies that $S = 0$ also. Therefore (1) already has the form

$$Rr + Pp + Qq + Zz = F, \tag{16}$$

containing only one second derivative. If $T \neq 0$, we have, since α is a double root of (6), $\alpha = -S/2R$; and from (5) and (7), together with this value of α, we find

$$S' = Su_x + 2Tu_y = (S\alpha + 2T)u_y$$
$$= \left(-\frac{S^2}{2R} + 2T\right)u_y = -\frac{S^2 - 4RT}{2R}u_y.$$

Hence, by (15), $S' = 0$. With $R' = S' = 0$, $T' = T$, and $v = y$, (4) then takes the form

$$T\frac{\partial^2 z}{\partial y^2} + P'\frac{\partial z}{\partial u} + Q'\frac{\partial z}{\partial y} + Zz = F. \tag{17}$$

This equation can sometimes be solved if $P' = 0$, for in this case it has the form (10) of Art. 67.

Example. Solve the equation

$$xr - ys - xyp + y^2q + yz = 1.$$

Solution. We first determine the roots of the quadratic equation (6). Since $R = x, S = -y, T = 0$, we have $x\theta^2 - y\theta = 0$, whence $\alpha = 0, \beta = y/x$. Hence (7) and (8) become

$$\frac{\partial u}{\partial x} = 0, \qquad \frac{\partial v}{\partial x} = \frac{y}{x}\frac{\partial v}{\partial y}.$$

From these we find the simple relations $u = y$, $v = xy$. We next determine, from (5), the coefficients of the transformed equation. We have, of course, $R' = 0$, $T' = 0$, and we also find $S' = -y^2 = -u^2$, $P' = y^2 = u^2$, $Q' = -y = -u$, $Z = y = u$. Consequently the new equation is

$$-u^2s' + u^2p' - uq' + uz = 1,$$

or

$$s' - p' + \frac{1}{u}q' - \frac{1}{u}z = -\frac{1}{u^2},$$

so that

$$L = -1, \quad M = \frac{1}{u}, \quad N = -\frac{1}{u}, \quad G = -\frac{1}{u^2}.$$

The transformed equation does not fall under any of the forms discussed in Art. 67, but we find that

$$N - L_u - LM = 0, \qquad N - M_v - LM = 0.$$

Therefore either equations (11) and (12), or equations (13) and (14), apply. Choosing the former pair, we get from (11),

$$\frac{\partial w}{\partial u} + \frac{1}{u}w = -\frac{1}{u^2},$$

$$uw = -\log u + f_1(v),$$

where $f_1(v)$ is arbitrary. Consequently (12) gives us

$$\frac{\partial z}{\partial v} - z = -\frac{\log u}{u} + \frac{1}{u}f_1(v),$$

$$e^{-v}z = \frac{\log u}{u}e^{-v} + \frac{1}{u}f_2(v) + g(u),$$

where $f_2(v)$ is a new arbitrary function such that $df_2/dv = e^{-v}f_1$, and $g(u)$ is also arbitrary. Finally, replacing u by y and v by xy, we obtain the solution

$$z = \frac{\log y}{y} + \frac{1}{y}f(xy) + e^{xy}g(y),$$

where $f(v) = e^v f_2(v)$.

The student should show that equations (13) and (14) lead to the same result.

69. Variation of parameters. In connection with linear ordinary differential equations, the method of variation of parameters (Art. 12) enables one to find a particular integral whenever the complementary function is known. A similar procedure applies to second order linear partial differential equations of a certain form. Only the general case will be discussed here; the argument is easily altered to take care of exceptional cases, for which the procedure to be described is also effective.*

We assume that the differential equation

$$Rr + Ss + Tt + Pp + Qq + Zz = F \tag{1}$$

has a complementary function of the form $f(u) + g(v)$, where u and v are known functionally independent expressions in x and y, so that the Jacobian $J \equiv u_x v_y - u_y v_x \neq 0$, and where f and g are arbitrary. Accordingly, $z = f(u) + g(v)$ must satisfy the homogeneous equation,

$$Rr + Ss + Tt + Pp + Qq + Zz = 0, \tag{2}$$

obtained from (1) by replacing the right member F by zero. Now we have, using accents to denote differentiation, as $f' = df/du$, $f'' = d^2f/du^2$, etc.,

$$p = u_x f' + v_x g', \qquad q = u_y f' + v_y g',$$

$$r = u_x^2 f'' + u_{xx} f' + v_x^2 g'' + v_{xx} g', \qquad t = u_y^2 f'' + u_{yy} f' + v_y^2 g'' + v_{yy} g',$$

$$s = u_x u_y f'' + u_{xy} f' + v_x v_y g'' + v_{xy} g'.$$

Substituting in (2) and rearranging, we get

$$(Ru_x^2 + Su_x u_y + Tu_y^2)f'' + (Ru_{xx} + Su_{xy} + Tu_{yy} + Pu_x + Qu_y)f' + Zf$$
$$+ (Rv_x^2 + Sv_x v_y + Tv_y^2)g'' + (Rv_{xx} + Sv_{xy} + Tv_{yy} + Pv_x + Qv_y)g' + Zg = 0.$$

Since this relation is to hold whatever the functions f and g, it follows that the coefficients of f, g, f', g', f'', and g'' in the above equation must all vanish separately. Hence, in particular, $Z = 0$, so that our differential equation must be of the form

$$Rr + Ss + Tt + Pp + Qq = F. \tag{3}$$

* A complete treatment may be found in the author's paper, "An Application of the Method of Parameters to Linear Partial Differential Equations", *Bull. Am. Math. Soc.*, February, 1930.

In addition, the relations

$$Ru_x^2 + Su_xu_y + Tu_y^2 = 0, \tag{4}$$

$$Rv_x^2 + Sv_xv_y + Tv_y^2 = 0, \tag{5}$$

$$Ru_{xx} + Su_{xy} + Tu_{yy} + Pu_x + Qu_y = 0, \tag{6}$$

$$Rv_{xx} + Sv_{xy} + Tv_{yy} + Pv_x + Qv_y = 0, \tag{7}$$

must be satisfied. Consequently the functions u and v must be such that u_x/u_y and v_x/v_y are roots of the quadratic equation

$$R\theta^2 + S\theta + T = 0, \tag{8}$$

by (4) and (5), and also such that they satisfy the homogeneous equation corresponding to (3), by (6) and (7). Since we have supposed u and v functionally independent, the roots $\alpha = u_x/u_y$ and $\beta = v_x/v_y$ of (8) will be different, for $\alpha = \beta$ leads to $u_xv_y - u_yv_x = J = 0$. Thus the discriminant $S^2 - 4RT$ of (8) must be different from zero.

Given an equation of the form (3), with $S^2 - 4RT \neq 0$, we therefore proceed as follows. We find the roots α and β of (8) and thus determine u and v, as in Laplace's method (Art. 68), from the Lagrange equations

$$u_x = \alpha u_y, \qquad v_x = \beta v_y. \tag{9}$$

If the functions u and v so found satisfy the homogeneous equation corresponding to (3), then $f(u) + g(v)$ will serve as complementary function of (3), and we have only to find a particular integral. For this purpose, we set

$$z = Uu + Vv + W, \tag{10}$$

where U, V, and W are unknown functions of x and y, to be employed as parameters. Then

$$p = Uu_x + Vv_x + U_xu + V_xv + W_x, \tag{11}$$

$$q = Uu_y + Vv_y + U_yu + V_yv + W_y. \tag{12}$$

When we say that (10) shall be a solution of (3), we impose one condition on our three parameters. We take as the two additional conditions,

$$U_xu + V_xv + W_x = 0, \tag{13}$$

$$U_yu + V_yv + W_y = 0, \tag{14}$$

so that (11) and (12) reduce and lead to

$$p = Uu_x + Vv_x, \qquad q = Uu_y + Vv_y,$$

$$r = Uu_{xx} + U_xu_x + Vv_{xx} + V_xv_x, \qquad t = Uu_{yy} + U_yu_y + Vv_{yy} + V_yv_y,$$

$$s = \frac{\partial p}{\partial y} = Uu_{xy} + U_yu_x + Vv_{xy} + V_yv_x \tag{15}$$

$$= \frac{\partial q}{\partial x} = Uu_{xy} + U_xu_y + Vv_{xy} + V_xv_y.$$

The two expressions for s give us the relation

$$U_yu_x + V_yv_x = U_xu_y + V_xv_y. \tag{16}$$

From (10), the total differential of z is

$$dz = U\,du + u\,dU + V\,dv + v\,dV + dW.$$

Now if we multiply (13) by dx and (14) by dy, and add, we get

$$u(U_x\,dx + U_y\,dy) + v(V_x\,dx + V_y\,dy) + W_x\,dx + W_y\,dy = 0,$$

or

$$u\,dU + v\,dV + dW = 0.$$

Therefore

$$dz = U\,du + V\,dv$$
$$= U(u_x\,dx + u_y\,dy) + V(v_x\,dx + v_y\,dy)$$
$$= (Uu_x + Vv_x)\,dx + (Uu_y + Vv_y)\,dy. \tag{17}$$

This expression is an exact differential, for

$$\frac{\partial}{\partial y}(Uu_x + Vv_x) = Uu_{xy} + U_yu_x + Vv_{xy} + V_yv_x,$$

$$\frac{\partial}{\partial x}(Uu_y + Vv_y) = Uu_{xy} + U_xu_y + Vv_{xy} + V_xv_y,$$

and these are equal by (16). Hence, if we find U and V, and insert their values in (17), we can integrate to get the desired particular integral.

Substituting from (15) into (3) (using $s = \partial q/\partial x$), we obtain

$$U(Ru_{xx} + Su_{xy} + Tu_{yy} + Pu_x + Qu_y) + (Ru_x + Su_y)U_x + Tu_yU_y$$
$$+V(Rv_{xx} + Sv_{xy} + Tv_{yy} + Pv_x + Qv_y) + (Rv_x + Sv_y)V_x + Tv_yV_y = F,$$

which, by (6) and (7), reduces to

$$(Ru_x + Su_y)U_x + Tu_yU_y + (Rv_x + Sv_y)V_x + Tv_yV_y = F. \tag{18}$$

We can eliminate V_x and V_y from (18) as follows. Multiply (16), on the left by Tv_y/v_x, and on the right by $-(Rv_x + Sv_y)/v_y$; these two quantities are equal by (5). Then we get

$$\frac{Tu_xv_y}{v_x} U_y + Tv_yV_y = -\frac{(Rv_x + Sv_y)u_y}{v_y} U_x - (Rv_x + Sv_y)V_x,$$

or

$$(Rv_x + Sv_y) V_x + Tv_yV_y = -\frac{(Rv_x + Sv_y)u_y}{v_y} U_x - \frac{Tu_xv_y}{v_x} U_y.$$

Substituting in (18), we find

$$\left(Ru_x + Su_y - \frac{Ru_yv_x}{v_y} - Su_y\right) U_x + \left(Tu_y - \frac{Tu_xv_y}{v_x}\right) U_y = F,$$

whence, using $J = u_xv_y - u_yv_x$, there is obtained

$$RJv_xU_x - TJv_yU_y = Fv_xv_y, \tag{19}$$

a Lagrange equation for the determination of U.

Similarly, using (4), multiply (16) on the left by Tu_y/u_x and on the right by $-(Ru_x + Su_y)/u_y$, to get

$$Tu_yU_y + \frac{Tu_yv_x}{u_x} V_y = -(Ru_x + Su_y) U_x - \frac{(Ru_x + Su_y)v_y}{u_y} V_x,$$

or

$$(Ru_x + Su_y) U_x + Tu_yU_y = -\frac{(Ru_x + Su_y)v_y}{u_y} V_x - \frac{Tu_yv_x}{u_x} V_y.$$

Substituting in (18), we eliminate U_x and U_y, thereby obtaining

$$\left(Rv_x + Sv_y - \frac{Ru_xv_y}{u_y} - Sv_y\right) V_x + \left(Tv_y - \frac{Tu_yv_x}{u_x}\right) V_y = F,$$

or

$$RJu_xV_x - TJu_yV_y = -Fu_xu_y, \tag{20}$$

a Lagrange equation in V.

Equations (19) and (20) are easily solved. The equations subsidiary to (19) are

$$\frac{dx}{RJv_x} = \frac{dy}{-TJv_y} = \frac{dU}{Fv_xv_y}. \tag{21}$$

If we multiply (4) by v_xv_y, and (5) by u_xu_y, and subtract, we get

$$R(u_x^2v_xv_y - u_xu_yv_x^2) - T(u_xu_yv_y^2 - u_y^2v_xv_y) = 0,$$

or

$$RJu_xv_x - TJu_yv_y = 0. \tag{22}$$

Hence u_x and u_y, used as multipliers (Art. 37) in the first two ratios of (21), yield $u_x\,dx + u_y\,dy = 0$, whence $u = a$, a constant. Then the first and third ratios in (21) give us

$$\frac{dU}{dx} = \frac{Fv_y}{RJ}. \tag{23}$$

Replacement of y, obtained from $u = a$ in terms of x and a, in (23), integration, and subsequent replacement of a by u, gives us a suitable expression for U.

Likewise, the equations subsidiary to (20) are

$$\frac{dx}{RJu_x} = \frac{dy}{-TJu_y} = \frac{dV}{-Fu_xu_y}. \tag{24}$$

Using v_x and v_y as multipliers, (22) gives us $v_x\,dx + v_y\,dy = 0$, whence $v = b$, a constant. Solving $v = b$ for y in terms of x and b, and inserting in

$$\frac{dV}{dx} = -\frac{Fu_y}{RJ}, \tag{25}$$

we are enabled to find V.

The values of U and V found from (23) and (25), which need not contain arbitrary functions if we are seeking merely a particular integral of (3), are then substituted in (17) and the integration performed. The sum of this particular integral and the complementary function $f(u) + g(v)$ then furnishes us with a solution of (3).

Example. Solve the equation

$$x(1 - x)r - y(1 - x^2)s + y^2(1 - x)t + p - xyq = 1.$$

Solution. We first form the quadratic equation (8); after division by the common factor $(1 - x)$, this is

$$x\theta^2 - y(1 + x)\theta + y^2 = 0.$$

This yields $\alpha = y/x$, $\beta = y$, whence equations (9) become

$$xu_x - yu_y = 0, \qquad v_x - yv_y = 0.$$

We readily find $u = xy$ and $v = ye^x$ as simple solutions of these equations, and it is also easy to verify that these functions satisfy the given differential equation. Hence

$$f(xy) + g(ye^x)$$

is a complementary function.

Equation (23) then gives us, using $xy = a$ and $J = (1 - x)ye^x$,

$$\frac{dU}{dx} = \frac{e^x}{x(1-x)^2ye^x} = \frac{1}{a(1-x)^2},$$

whence we may take

$$U = \frac{1}{a(1-x)} = \frac{1}{xy(1-x)}.$$

Similarly, (25) becomes, with the aid of $ye^x = b$,

$$\frac{dV}{dx} = -\frac{x}{x(1-x)^2ye^x} = -\frac{1}{b(1-x)^2},$$

so that

$$V = -\frac{1}{b(1-x)} = -\frac{1}{(1-x)ye^x}.$$

Substituting in (17), we get

$$dz = \left[\frac{1}{x(1-x)} - \frac{1}{1-x}\right]dx + \left[\frac{1}{y(1-x)} - \frac{1}{(1-x)y}\right]dy = \frac{dx}{x},$$

and consequently

$$z = \log x$$

is a particular integral. Therefore the given differential equation has the solution

$$z = f(xy) + g(ye^x) + \log x.$$

Application of the Laplace transformation to equation (3) yields the simple equation

$$\frac{\partial^2 z}{\partial u \partial v} = -\frac{Fu_yv_y}{RJ^2}, \tag{26}$$

and it may also be shown * that our results are closely related to Laplace's method. However, practical difficulties sometimes prevent the use of Laplace's method directly. Thus, for the above example, (26) takes the form

$$\frac{\partial^2 z}{\partial u \partial v} = -\frac{1}{(1-x)^3y^2e^x}.$$

* See the paper previously cited.

To treat this, it would be necessary to replace x and y in the right member by their values in terms of u and v. But the inverse of the transformation $u = xy$, $v = ye^x$ cannot be obtained in elementary finite form, and consequently we are unable to complete the process.

EXERCISES

Solve each of the equations in Exercises 1–10.

1. $x^2r + xys + xp + yq - z = 0.$
2. $2yr + s + (2y - 1)p + q - z = 2ye^y.$
3. $y(x + y)(r - s) - xp - yq - z = 4xy + y^2.$
4. $x^2r - 3xys + 2y^2t + 2xp + yq = 6x^2.$
5. $x^2r - 2xys + y^2t + (2x - x^2y^2)p + xy^3q - 2x^2y^4z = y^2e^{xy}.$
6. $x(y - x)r + (x^2 - y^2)s + y(y - x)t + (y + x)(p - q) = 6x(y - x)^2.$
7. $y(xy - 1)r - (x^2y^2 - 1)s + x(xy - 1)t + (x - y^2)p + (y - x^2)q = 3(xy - 1)^2.$
8. $(y - 1)r - (y^2 - 1)s + y(y - 1)t + p - q = 2(y - 1)^2e^{-x}.$
9. $x(xy - 1)r - (x^2y^2 - 1)s + y(xy - 1)t + (x - 1)p + (y - 1)q = (xy - 1)^2.$
10. $x(x + y)r + (x^2 - y^2)s - y(x + y)t + (y - x)(p + q) = 16x(x + y).$

11. In Laplace's method, if $N - L_u - LM \neq 0$ and $N - M_v - LM \neq 0$, we may proceed as follows. Let

$$w = \frac{\partial z}{\partial v} + Lz, \qquad K = N - L_u - LM \neq 0,$$

so that equation (10), Art. 68, becomes

$$\frac{\partial w}{\partial u} + Mw + Kz = G.$$

By solving this equation for z, and substituting in $w = \partial z/\partial v + Lz$, show that a new equation of the form

$$\frac{\partial^2 w}{\partial u \partial v} + L' \frac{\partial w}{\partial u} + M' \frac{\partial w}{\partial v} + N'w = G'$$

is obtained. This is of the same type as (9), and hence may be similarly treated.

12. Alternatively to the procedure of Exercise 11, we may set $w = \partial z/\partial u + Mz$, $K = N - M_v - LM \neq 0$. Show that this also leads to an equation for w of the type (9), Art. 68. Each of these transformation processes may be repeated until the necessary condition for solvability is satisfied.

13. Show that we may use the quadratic equation $T\theta^2 + S\theta + R = 0$ in the place of equation (6), Art. 68, and outline the procedure analogous to that described. This alternative viewpoint is useful when $R = 0$.

14. Using the method of Exercise 13, solve the equation

$$xys - y^2t + xp - 2yq = 2x^2y.$$

15. If, in Art. 68, $S^2 - 4RT = 0$, we may alternatively take $u = x$ and determine v from equation (8). Show that this also leads to a new equation containing only one second derivative, and find the condition under which it is solvable.

16. Using the method of Exercise 15, solve the equation

$$x^2r - 2x^2s + x^2t + xp - xq - z = y - x.$$

17. Derive equation (26), Art. 69.

18. Show that the right member of equation (26), Art. 69, may also be expressed as $-Fu_xv_x/TJ^2$.

19. Show that, for the method of Art. 69,

$$u_x = Jy_v, \qquad u_y = -Jx_v, \qquad v_x = -Jy_u, \qquad v_y = Jx_u.$$

20. Using the relations of Exercise 19, show that equation (26), Art. 69, leads to

$$\frac{\partial z}{\partial u} = \int_v \frac{Fv_y}{RJ}\frac{\partial x}{\partial v}\,dv + f_1(u),$$

$$\frac{\partial z}{\partial v} = -\int_u \frac{Fu_y}{RJ}\frac{\partial x}{\partial u}\,du + g_1(v),$$

where a subscript on an integral sign denotes the variable with respect to which the partial integration is to be performed, and f_1 and g_1 are arbitrary functions.

70. Geometric problems. In this article we shall illustrate, by means of examples, a few geometric problems involving linear partial differential equations of order higher than the first.

Example 1. Find the equation of the surface satisfying the differential equation $r = 0$ and containing the lines

$$x + y = 1, \quad z = 0 \qquad \text{and} \qquad z = 2, \quad x = 0. \tag{1}$$

Solution. We readily find, by the method of either Art. 62 or Art. 67, the solution

$$z = f(y) + xg(y) \tag{2}$$

of the given differential equation. We have now to determine the nature of the functions $f(y)$ and $g(y)$, which are arbitrary in (2), so that the given geometric conditions shall be fulfilled.

For any point on the line $x + y = 1$, $z = 0$, we must have $y = 1 - x$ and $z = 0$. Inserting these values in (2), we get the relation

$$0 = f(1 - x) + xg(1 - x), \tag{3}$$

which must be satisfied by f and g identically in x. Similarly, for any point of the second of the lines (1), we must have $z = 2$, $x = 0$, whence insertion in (2) gives us

$$2 = f(y). \tag{4}$$

The f-function is therefore determined as the constant 2. Setting $f(1 - x) = 2$ in (3), we then get

$$g(1 - x) = -\frac{2}{x}. \tag{5}$$

Now since the g-function in the left member of (5) has the argument $1 - x$, we express the right member in terms of this quantity. Evidently, then,

$$g(1 - x) = \frac{2}{(1 - x) - 1},$$

or

$$g(y) = \frac{2}{y - 1}. \tag{6}$$

Substituting from (4) and (6) into (2), we therefore have

$$z = 2 + \frac{2x}{y - 1} \tag{7}$$

as the required surface. Equation (7) represents a hyperbolic paraboloid generated by a line moving always parallel to the xz-plane and passing through the lines (1).

Example 2. Find the equation of the surface satisfying the differential equation $s - 2t = 0$ and tangent to the paraboloid $z = x^2 - 2y^2$ along the section by the plane $y = 2x - 2$.

Solution. The given differential equation is easily solved (Art. 62 or Art. 67), and we find the solution

$$z = f(x) + g(y + 2x). \tag{8}$$

If the required surface, included in the family (8), is to be tangent to the hyperbolic paraboloid $z = x^2 - 2y^2$ along a plane section, the values of p for these two surfaces must be the same at any point of that section, and the corresponding values of q must coincide there. For the surfaces (8), we get

$$p = f'(x) + 2g'(y + 2x), \qquad q = g'(y + 2x), \tag{9}$$

and for the surface $z = x^2 - 2y^2$,

$$p = 2x, \qquad q = -4y. \tag{10}$$

Using $y = 2x - 2$, which must hold at each point of tangency, we then have from (9) and (10),

$$2x = f'(x) + 2g'(4x - 2), \tag{11}$$

$$8 - 8x = g'(4x - 2). \tag{12}$$

Relation (12) may be written as

$$g'(4x - 2) = -2(4x - 2) + 4,$$

whence we get, by integration,

$$g(4x - 2) = -(4x - 2)^2 + 4(4x - 2) + a, \tag{13}$$

where a is a constant. Combining (11) and (12), we find

$$f'(x) = 2x - 2(8 - 8x) = 18x - 16,$$

and consequently

$$f(x) = 9x^2 - 16x + b, \tag{14}$$

where b is a constant. Changing the argument $4x - 2$ in (13) to $y + 2x$, and inserting the result, together with (14), in (8), we have

$$\begin{aligned} z &= 9x^2 - 16x - (y + 2x)^2 + 4(y + 2x) + c \\ &= 5x^2 - 4xy - y^2 - 8x + 4y + c, \end{aligned} \tag{15}$$

where $c = a + b$. To determine the value of c, we may use the fact that any conveniently chosen point on the section of tangency must lie on the surface (15). Taking $x = 1$, the plane equation $y = 2x - 2$ gives us $y = 0$, and the surface equation $z = x^2 - 2y^2$ then yields $z = 1$. The coordinates (1, 0, 1), inserted in (15), give us $c = 4$, and therefore the required surface is the quadric surface

$$z = 5x^2 - 4xy - y^2 - 8x + 4y + 4. \tag{16}$$

Example 3. Show that the differential equation

$$(x^3D_x^3 + 3x^2yD_x^2D_y + 3xy^2D_xD_y^2 + y^3D_y^3)z = 0 \tag{17}$$

is satisfied by surfaces all of whose sections by planes through the z-axis are straight lines and parabolas.

Solution. We may solve equation (17) by the method of Art. 66. Letting

$$x = e^u, \qquad y = e^v, \tag{18}$$

(17) becomes

$$[D_u(D_u - 1)(D_u - 2) + 3D_u(D_u - 1)D_v + 3D_uD_v(D_v - 1) + D_v(D_v - 1)(D_v - 2)]z = 0.$$

This reduces to

$$[(D_u + D_v)^3 - 3(D_u + D_v)^2 + 2(D_u + D_v)]z = 0,$$

or

$$(D_u + D_v)(D_u + D_v - 1)(D_u + D_v - 2)z = 0.$$

Hence we have the solution (Art. 62)

$$z = f_1(v - u) + e^u g_1(v - u) + e^{2u} h_1(v - u).$$

Replacing u by $\log x$ and v by $\log y$, we see that (17) has the solution

$$z = f\left(\frac{y}{x}\right) + xg\left(\frac{y}{x}\right) + x^2h\left(\frac{y}{x}\right). \tag{19}$$

Now every plane through the z-axis, except $x = 0$, has an equation of the form $y = mx$. Substituting $y = mx$ in (19), we get as the equation of the sections in these planes,

$$z = f(m) + xg(m) + x^2h(m). \tag{20}$$

When $h(m) = 0$, (20) represents a straight line in the plane $y = mx$, and when $h(m) \neq 0$, we have a parabolic section. To take care of the exceptional plane, $x = 0$, we write the solution of (17) in the alternative form,

$$z = f\left(\frac{x}{y}\right) + yg\left(\frac{x}{y}\right) + y^2h\left(\frac{x}{y}\right). \tag{19'}$$

Evidently all these surfaces have straight line or parabolic sections by the plane $x = 0$. Therefore every solution of the forms (19) and (19′) whatever the functions f, g, and h, represents a surface with the stated geometric property.

EXERCISES

1. Find the equation of the surface satisfying the differential equation $r + 2s + t = 0$ and containing the lines $y = 0, z = 0$ and $y = 3x, z = 6x$.

2. Find the equation of the surface satisfying the differential equation $r - 6s + 9t = 0$ and passing through the line $x = 0, z = y$ and the parabola $y = 0, z = 3x^2$.

3. Find the equation of the surface satisfying the differential equation $r - 3s + 2t = 8$ and passing through the y-axis and the parabola $x + y = 0, z = 4x^2$.

4. Find the equation of the surface satisfying the differential equation $r + s - 2t = 0$ and tangent to the paraboloid $z = 2x^2 + 2y^2 - 1$ along its section by the plane $y = x$.

5. Find the equation of the surface satisfying the differential equation $r - 2s = 4$ and tangent to the surface $z = x^2 - 6xy + 8y^2$ along its section by the plane $x + y = 1$.

6. Show that there are infinitely many surfaces whose equations satisfy the differential equation $x^2r - y^2t + xp - yq = 0$ and which are tangent to the surface $z = 2x^2y^2$ along its section by the plane $y = x$.

7. Find the equation of the two-parameter family of planes satisfying the differential equation $ys + p = 3$.

8. Find the equations of the families of quadric surfaces satisfying the differential equation $s - 2xt = 0$.

9. Find the equation of the quadric surface satisfying the differential equation $s \cos x + q \sin x = 0$ and passing through the y-axis, the line $x = 1, z = 1$, and the line $x = 2, z = 6$.

10. Find the equation of the quadric surface satisfying the differential equation $xyr + x^2s - yp = 0$ and passing through the parabolas $z = x^2 - 3$, $y = 0$ and $z = 2y^2 - 3$, $x = 0$.

11. Find the equation of the quadric cone satisfying the differential equation $yt - q = 0$, having its vertex at the origin, and passing through the parabola $z = y^2 - 4, x = 1$.

12. Show that the differential equation $x^2r + 2xys + y^2t = 0$ is satisfied by ruled surfaces generated by straight lines passing through the z-axis.

13. Find the equation of the surface satisfying the differential equation $r + t = 0$ and tangent to the surface $y^2 - z^2 = 1$ along its section by the plane $x = 0$.

14. Find the equation of the surface satisfying the differential equation $(D_x^3 - 3D_x^2D_y + 3D_xD_y^2 - D_y^3)z = 0$ and containing the lines: $z = y$, $x = 0$; $z = 2y - 1$, $x = 1$; and $z = 3y - 2$, $x = 2$.

15. Find the equation of the surface satisfying the differential equation $(D_x^3 + 6D_x^2D_y + 12D_xD_y^2 + 8D_y^3)z = 0$ and containing the lines: $z = 6x + 1$, $y = 0$; $z = 2x - 2$, $y = 1$; and $z = 10x + 4$, $y = -1$.

16. Using the method of Example 2, Art. 63, find the equation of a quadric surface satisfying the differential equation $3t - p = 0$ and passing through the curves $z = 2 + 24x$, $y = 0$ and $z = 2 - 3y + 4y^2$, $x = 0$.

17. Find the equation of the quadric surface satisfying the differential equation $2r - 3p + q = 0$ and passing through the points (0, 0, 0), (−1, 1, 8), and (2, −1, 11).

18. Using the method of Art. 64, find the equation of the surface satisfying the differential equation $r - q = 0$ and passing through the y-axis and the curve $z = 2 \sin x$, $y = 0$.

19. Find the equation of the surface satisfying the differential equation $r - t = 0$ and passing through the curves $z = \sin x + \cos x$, $y = 0$ and $z = \cos 2x$, $y = x$.

20. If $f(x + iy) = \phi(x, y) + i\psi(x, y)$, where x, y, ϕ, and ψ are all real and $i = \sqrt{-1}$, show that $z = \phi(x, y)$ and $z = \psi(x, y)$ both satisfy Laplace's equation, $r + t = 0$, and that the two families of curves $\phi(x, y) = a$ and $\psi(x, y) = b$ are orthogonal trajectories of each other.

71. The vibrating string. In Art. 30 we derived the partial differential equation of the vibrating string,

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}, \tag{1}$$

where x (ft.) is the distance measured along the stretched string in its equilibrium position, t (sec.) is time, $y = y(x, t)$ (ft.) is the displacement from equilibrium position, and $a = \sqrt{Fg/w}$ (ft./sec.) is a constant; F (lb.) is the constant tension, w (lb./ft.) is the linear density of the string, and $g = 32.2$ ft./sec.^{2}

We assumed, in Art. 30, that the two fixed points of support were at (0, 0) and (L, 0), where L (ft.) is the length of the string. We now take these physical facts and express them as *boundary conditions.* Moreover, we shall stipulate typical *initial conditions*, that is, the displacement and velocity of each point of the string at the time $t = 0$. When a physical problem is thus completely specified, by a combination of the intrinsic equation (1) and initial and boundary conditions, we may expect to find a unique mathematical solution giving us the displacement of each point of the string at each instant. The procedure of finding such a solution, using the method of separation of variables (Art. 64), is illustrated in the following example.

Example. Find the displacement function $y(x, t)$ which satisfies the differential equation (1), the boundary conditions

$$y(0, t) = 0, \qquad t > 0, \tag{2}$$

$$y(L, t) = 0, \qquad t > 0, \tag{3}$$

and the initial conditions

$$y(x, 0) = m(Lx - x^2), \qquad 0 < x < L, \tag{4}$$

$$\left.\frac{\partial y}{\partial t}\right]_{t=0} = 0, \qquad 0 < x < L, \tag{5}$$

where m (ft.$^{-1}$) is a constant, numerically small.

Solution. As in Art. 64, we assume a solution of (1) in the form

$$y(x, t) = X(x) \cdot T(t), \tag{6}$$

which, when substituted in (1), leads to

$$XT'' = a^2X''T,$$

$$\frac{X''}{X} = \frac{T''}{a^2T} = k,$$

a constant. We thus obtain three types of solutions:

$$y = (c_1e^{\sqrt{k}x} + c_2e^{-\sqrt{k}x})(c_3e^{a\sqrt{k}t} + c_4e^{-a\sqrt{k}t}), \qquad k > 0; \tag{7}$$

$$y = (c_5 \sin \sqrt{-k}x + c_6 \cos \sqrt{-k}x)(c_7 \sin a\sqrt{-k}t + c_8 \cos a\sqrt{-k}t), \qquad k<0; \tag{8}$$

$$y = (c_9x + c_{10})(c_{11}t + c_{12}), \qquad k = 0. \tag{9}$$

Supposing no energy is dissipated, so that each position of the string is repeated indefinitely often, we see that the motion must be periodic, and consequently only form (8) is physically suitable. Using this, we now apply the boundary conditions (2) and (3), which express the fact that the ends of the string undergo no displacement. Substituting $x = 0$, $y = 0$ in (8), we get the relation

$$c_6(c_7 \sin a\sqrt{-k}t + c_8 \cos a\sqrt{-k}t) = 0, \tag{10}$$

which must hold for every $t > 0$. Evidently this condition is satisfied if we take $c_6 = 0$, whence (8) reduces to

$$y = c_5 \sin \sqrt{-k}x\ (c_7 \sin a\sqrt{-k}t + c_8 \cos a\sqrt{-k}t), \qquad k < 0. \tag{11}$$

From (3) we then get the relation

$$c_5 \sin \sqrt{-k}L\ (c_7 \sin a\sqrt{-k}t + c_8 \cos a\sqrt{-k}t) = 0, \tag{12}$$

3. If the string is initially at rest in its equilibrium position and each of its points is given the velocity

$$\left.\frac{\partial y}{\partial t}\right]_{t=0} = v_0 \sin^3 \frac{\pi x}{L}, \qquad 0 < x < L,$$

where v_0 (ft./sec.) is a small positive constant, find $y(x, t)$.

4. Using the data of Exercise 1, find the maximum permissible value of v_0 in Exercise 3 if the displacement of the midpoint of the string should not exceed 1 in.

5. If the initial configuration of the string is given by

$$y(x, 0) = y_0 \sin^5 \frac{\pi x}{L}, \qquad 0 < x < L,$$

where y_0 (ft.) is a small positive constant, and the string is released from rest in this position, find $y(x, t)$.

6. Using the data of Exercise 1 for the string of Exercise 5, and taking $y_0 = 1$ in., find $y(1, 0.1)$.

7. If the string is initially at rest in its equilibrium position, and each of its points is given the velocity

$$\left.\frac{\partial y}{\partial t}\right]_{t=0} = K(Lx - x^2), \qquad 0 < x < L,$$

where K (ft.$^{-1}$ sec.$^{-1}$) is a sufficiently small positive constant, find $y(x, t)$.

8. Using the data of Exercise 1 for the string of Exercise 7, and taking $K = 2$ (ft.$^{-1}$ sec.$^{-1}$), find the displacement of the midpoint of the string when $t = 0.1$ sec.

9. If the string is initially at rest in its equilibrium position, and each of its points is given the velocity

$$\left.\frac{\partial y}{\partial t}\right]_{t=0} = cx, \qquad 0 < x \leqq \frac{L}{2},$$

$$\left.\frac{\partial y}{\partial t}\right]_{t=0} = c(L - x), \qquad \frac{L}{2} < x < L,$$

where c (sec.$^{-1}$) is a sufficiently small positive constant, find $y(x, t)$.

10. Using the data of Exercise 1 for the string of Exercise 9, and taking $c = 3$ (sec.$^{-1}$), find the displacement of the midpoint of the string when $t = 0.1$ sec.

72. One-dimensional heat flow. In Art. 31 we derived the partial differential equation of one-dimensional heat flow,

$$\frac{\partial \tau}{\partial t} = \alpha^2 \frac{\partial^2 \tau}{\partial x^2}, \tag{1}$$

where x (cm.) is the distance measured along the direction of flow, t (sec.) is time, $\tau = \tau(x, t)$ (° C.) is the temperature, and $\alpha^2 = K/c\delta$ (cm.^{2}/sec.) is the diffusivity, a constant; K (cal./cm. deg. sec.) is the thermal conductivity, c (cal./gr. deg.) is the specific heat, and δ (gr./cm.^{3}) is the density, of the material of the bar in which the heat flow takes place.

We consider now a typical physical problem involving one-dimensional heat flow.

Example. A bar, 10 cm. long, with insulated sides, has its ends A and B kept at 20° and 40°, respectively, until steady-state conditions prevail, that is, until the temperature at any interior point no longer changes with time. The temperature at A is then suddenly raised to 50° and at the same instant that at B is lowered to 10°. Find the subsequent temperature function $\tau(x, t)$.

Solution. From the verbal statement of the problem we first mathematically formulate boundary and initial conditions. For our boundary conditions we clearly have, taking $x = 0$ at the end A,

$$\tau(0, t) = 50, \qquad t > 0, \tag{2}$$

$$\tau(10, t) = 10, \qquad t > 0. \tag{3}$$

To obtain an equation expressing the initial condition, we proceed as follows. Previous to the temperature changes at the ends, at $t = 0$, the heat flow was independent of time. But if the temperature τ depends only upon x and not upon t, equation (1) reduces to

$$\frac{d^2\tau}{dx^2} = 0, \tag{4}$$

the solution of which is evidently

$$\tau = ax + b, \tag{5}$$

where a and b are arbitrary constants. Since $\tau = 20$ for $x = 0$, and $\tau = 40$ for $x = 10$, we get from (5),

$$20 = b, \qquad 40 = 10a + b,$$

whence $a = 2$ and $b = 20$. Thus our initial condition is expressed by

$$\tau(x, 0) = 2x + 20, \qquad 0 < x < 10. \tag{6}$$

We have therefore to find a temperature function $\tau(x, t)$ satisfying the differential equation (1), the boundary conditions (2) and (3), and the initial condition (6).

Now if we immediately follow the procedure adopted in Art. 71, we encounter a difficulty that did not arise there. The boundary values in our vibrating string problem were both zero, and consequently we were able to find, not only one solution of the differential equation, satisfying these boundary conditions, but the necessary more general solution made up of the sum of such solutions. Here, however, we have non-zero boundary values, and therefore we modify the procedure.

We break up the required function $\tau(x, t)$ into two parts,

$$\tau(x, t) = \tau_s(x) + \tau_t(x, t), \tag{7}$$

where $\tau_s(x)$ is a solution of (1), involving only x and satisfying the boundary conditions (2) and (3), and $\tau_t(x, t)$ is a function defined by (7). Thus $\tau_s(x)$ is a *steady-state* solution, of the form (5), and $\tau_t(x, t)$ may then be regarded as a *transient* solution, numerically decreasing with increase of t so as to become negligible after a sufficient lapse of time.

Using the form (5) for $\tau_s(x)$, we get, since $\tau_s(0) = 50$ and $\tau_s(10) = 10$,

$$\tau_s(x) = 50 - 4x. \tag{8}$$

Consequently we have, from (7), (2), and (8),

$$\tau_t(0, t) = \tau(0, t) - \tau_s(0) = 50 - 50 = 0; \tag{9}$$

from (7), (3), and (8),

$$\tau_t(10, t) = \tau(10, t) - \tau_s(10) = 10 - 10 = 0; \tag{10}$$

and from (7), (6), and (8),

$$\tau_t(x, 0) = \tau(x, 0) - \tau_s(x) = 2x + 20 - 50 + 4x = 6x - 30. \tag{11}$$

Equations (9)–(11) express the boundary and initial conditions relative to the transient solution. Since the boundary values given by (9) and (10) are both zero, we may follow the method used in Art. 71 to determine $\tau_t(x, t)$.

We assume $\tau_t(x, t)$ to be of the form

$$\tau_t(x, t) = X(x) \cdot T(t), \tag{12}$$

substitute in (1), and separate the variables. This gives us

$$\begin{aligned} XT' &= \alpha^2 X''T, \\ \frac{T'}{\alpha^2 T} &= \frac{X''}{X} = k, \end{aligned} \tag{13}$$

where k is a constant. We then find three types of solutions:

$$\tau_t = e^{\alpha^2 kt}(c_1 e^{\sqrt{k}x} + c_2 e^{-\sqrt{k}x}), \qquad k > 0, \tag{14}$$

$$\tau_t = e^{\alpha^2 kt}(c_3 \sin \sqrt{-k}x + c_4 \cos \sqrt{-k}x), \qquad k < 0, \tag{15}$$

$$\tau_t = c_5 x + c_6, \qquad k = 0. \tag{16}$$

Since τ_t is to decrease numerically with increase of t, form (15) must be chosen. From (9) we get the relation

$$0 = c_4 e^{\alpha^2 kt}, \qquad t > 0,$$

which can be met only by taking $c_4 = 0$. Condition (10) gives us

$$0 = c_3 e^{\alpha^2 kt} \sin 10\sqrt{-k}, \qquad t > 0,$$

and since c_3 cannot be zero, else τ_t would be identically zero, we take $10\sqrt{-k} = n\pi$, where n is an integer. Hence (15) reduces to

$$\tau_t = c_3 e^{-n^2\pi^2\alpha^2 t/100} \sin \frac{n\pi x}{10}. \tag{17}$$

It is apparent that no single term of the form (17) will satisfy condition (11). Reasoning as in Art. 71, we are thus led to the infinite series

$$\tau_t = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2\alpha^2 t/100} \sin \frac{n\pi x}{10}. \tag{18}$$

This series will formally satisfy (1), (9), and (10). Relation (11) now gives us

$$6x - 30 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10}, \qquad 0 < x < 10. \tag{19}$$

We easily get the needed Fourier half-range sine series for $6x - 30$ by combining the series (I) and (II) of Art. 60. Setting $L = 10$ in these series, we find

$$\begin{aligned} 6x - 30 &= \frac{120}{\pi}\left(\sin\frac{\pi x}{10} - \frac{1}{2}\sin\frac{2\pi x}{10} + \frac{1}{3}\sin\frac{3\pi x}{10} - \cdots\right) \\ &\quad - \frac{120}{\pi}\left(\sin\frac{\pi x}{10} + \frac{1}{3}\sin\frac{3\pi x}{10} + \cdots\right) \\ &= -\frac{60}{\pi}\left(\sin\frac{\pi x}{5} + \frac{1}{2}\sin\frac{2\pi x}{5} + \frac{1}{3}\sin\frac{3\pi x}{5} + \cdots\right). \end{aligned} \tag{20}$$

Therefore

$$b_1 = 0, \quad b_2 = -\frac{60}{\pi}, \quad b_3 = 0, \quad b_4 = -\frac{60}{2\pi}, \quad \cdots,$$

and substitution in (18) yields

$$\tau_t = -\frac{60}{\pi}\left(e^{-4\pi^2\alpha^2 t/100}\sin\frac{\pi x}{5} + \frac{1}{2}e^{-16\pi^2\alpha^2 t/100}\sin\frac{2\pi x}{5} + \cdots\right). \tag{21}$$

Finally, combining (8) and (21) into (7), we have the required solution,

$$\tau(x, t) = 50 - 4x - \frac{60}{\pi}\sum_{n=1}^{\infty}\frac{1}{n} e^{-4n^2\pi^2\alpha^2 t/100} \sin\frac{n\pi x}{5}. \tag{22}$$

This series converges for $0 < x < 10$, $t > 0$. Except when t is small, the presence of the exponential functions make convergence rapid, so that usually only a few terms need be used in computation.

EXERCISES

1. Show that the temperature at the midpoint of the bar in the above example, originally 30° when $t = 0$, remains the same for all $t > 0$, regardless of the material of the bar.

2. If the bar in the above example is of copper, with $K = 0.84$ cal./cm. deg. sec., $c = 0.092$ cal./gr. deg., and $\delta = 8.81$ gr./cm.^{3}, find the temperature at a point 2.5 cm. from A 10 sec. after the change.

3. A bar, 30 cm. long, has its ends A and B kept at 20° C. and 80° C., respectively, until steady-state conditions prevail. The temperature at each end is then suddenly reduced to 0° C. and kept so. Find the resulting temperature function $\tau(x, t)$, taking $x = 0$ at the end A.

4. If the bar in Exercise 3 is of steel, with $K = 0.086$ cal./cm. deg. sec., $c = 0.118$ cal./gr. deg., and $\delta = 7.70$ gr./cm.^{3}, find the temperature at the midpoint of the bar 10 min. after the change.

5. A bar, 40 cm. long, originally has a temperature of 0° C. throughout its length. At time $t = 0$, the temperature at one end ($x = 0$) is raised to 50° and that at the other is raised to 100°. Find the resulting temperature function $\tau(x, t)$.

6. If the bar in Exercise 5 is of copper, with $K = 0.84$ cal./cm. deg. sec., $c = 0.092$ cal./gr. deg., and $\delta = 8.81$ gr./cm.^{3}, find the temperature at the midpoint of the bar 1 min. after the change.

7. Find the time required for the temperature of the midpoint of the bar of Exercise 6 to reach 90 per cent of its steady-state value.

8. The temperature at one end ($x = 0$) of a bar 50 cm. long is kept at 0° C. and that at the other end is kept at 100° C. until steady-state conditions prevail. The two ends are then suddenly insulated, so that the temperature gradient, $\partial\tau/\partial x$, is zero at each end thereafter. Find the temperature function $\tau(x, t)$.

9. Show that the temperature at the center of the bar of Exercise 8 remains 50° C., and that the sum of the temperatures at any two points equidistant from the center is 100°.

10. If the bar in Exercise 8 is of steel, with $K = 0.086$ cal./cm. deg. sec., $c = 0.118$ cal./gr. deg., and $\delta = 7.70$ gr./cm.^{3}, find the time required for the temperature of a point 10 cm. from the cold end to reach 90 per cent of its steady-state value.

73. Two-dimensional heat flow. In Art. 32 we derived the linear partial differential equation for heat flow in space, using orthogonal curvilinear coordinates. We shall confine ourselves here to two-dimensional steady-state heat flow. Using rectangular cordinates (x, y), we then have to deal with Laplace's equation in the form

$$\frac{\partial^2\tau}{\partial x^2} + \frac{\partial^2\tau}{\partial y^2} = 0 \tag{1}$$

(cf. equation (9), Art. 32); and in polar coordinates (ρ, θ), Laplace's equation becomes

$$\rho^2\frac{\partial^2\tau}{\partial\rho^2} + \rho\frac{\partial\tau}{\partial\rho} + \frac{\partial^2\tau}{\partial\theta^2} = 0 \tag{2}$$

(cf. Exercise 1, Art. 32).

The method of separation of variables (Art. 64) may often be used to find the temperature function, $\tau(x, y)$ or $\tau(\rho, \theta)$, satisfying Laplace's equation and the accompanying boundary conditions. When the plate (assumed to have insulated faces to insure two-dimensional flow) is rectangular in shape, we naturally use Laplace's equation in the form (1), but if the plate has some circular arcs as boundaries, equation (2) may be preferable. We illustrate the latter situation in the following example.

Example. A semicircular plate of radius 10 cm. has insulated faces and heat flows in plane curves. The bounding diameter is kept at 0° C., so that we have, measuring θ from one of the bounding radii,

$$\tau(\rho, 0) = 0, \qquad 0 \leqq \rho < 10, \tag{3}$$

$$\tau(\rho, \pi) = 0, \qquad 0 < \rho < 10. \tag{4}$$

On the semicircumference, the maintained temperature distribution is given by

$$\begin{aligned} \tau(10, \theta) &= \frac{200}{\pi}\theta, & 0 \leqq \theta \leqq \frac{\pi}{2}, \\ \tau(10, \theta) &= \frac{200}{\pi}(\pi - \theta), & \frac{\pi}{2} \leqq \theta \leqq \pi. \end{aligned} \tag{5}$$

Thus the temperature increases linearly with θ from each end, attaining a maximum value of 100° for $\theta = \pi/2$. After steady-state conditions prevail, what is the temperature function $\tau(\rho, \theta)$?

Solution. We have to find a solution of the differential equation (2) meeting the boundary conditions (3)–(5). Assuming a solution of the form

$$\tau(\rho, \theta) = R(\rho) \cdot T(\theta), \tag{6}$$

we get by substitution in (2),

$$\rho^2 R''T + \rho R'T + RT'' = 0,$$

whence

$$\frac{\rho^2 R'' + \rho R'}{R} = -\frac{T''}{T} = k, \tag{7}$$

a constant. Three types of solutions for T evidently are

$$T = c_1 \sin \sqrt{k}\theta + c_2 \cos \sqrt{k}\theta, \qquad k > 0, \tag{8}$$

$$T = c_3 e^{\sqrt{-k}\theta} + c_4 e^{-\sqrt{-k}\theta}, \qquad k < 0, \tag{9}$$

$$T = c_5\theta + c_6, \qquad k = 0. \tag{10}$$

Since τ, and therefore T, must be a periodic function of θ, rather than an indefinitely increasing or decreasing function of θ, we must choose $k > 0$.

The corresponding equation for R,

$$\rho^2 \frac{d^2R}{d\rho^2} + \rho \frac{dR}{d\rho} - kR = 0, \tag{11}$$

is an Euler differential equation (Art. 13). If we set $\rho = e^z$, (11) is transformed into

$$\frac{d^2R}{dz^2} - kR = 0.$$

With $k > 0$, this equation has the solution

$$R = c_7 e^{\sqrt{k}z} + c_8 e^{-\sqrt{k}z} = c_7 \rho^{\sqrt{k}} + c_8 \rho^{-\sqrt{k}}. \tag{12}$$

Thus we have, combining (8) and (12) into (6),

$$\tau = (c_7 \rho^{\sqrt{k}} + c_8 \rho^{-\sqrt{k}})\ (c_1 \sin \sqrt{k}\theta + c_2 \cos \sqrt{k}\theta). \tag{13}$$

Now condition (3) tells us that, in particular, $\tau(0, 0) = 0$. Hence c_8 must be taken as zero in (13). Applying (3) in full, we then get the relation

$$0 = c_2 c_7 \rho^{\sqrt{k}}, \qquad 0 \leqq \rho < 10. \tag{14}$$

As τ is not to be identically zero, we cannot take $c_7 = 0$, but (14) may be met by setting $c_2 = 0$. Next applying (4), we find

$$0 = c_1 c_7 \rho^{\sqrt{k}} \sin \sqrt{k}\pi, \qquad 0 < \rho < 10, \tag{15}$$

which (since c_1 and c_7 must both be different from zero) we meet by choosing $\sqrt{k} = n$, an integer. Thus we have, so far,

$$\tau = c_1 c_7 \rho^n \sin n\theta. \tag{16}$$

A single term of the form (16), or a sum of a finite number of such terms, will be inadequate to meet boundary condition (5). We therefore take

$$\tau = \sum_{n=1}^{\infty} b_n \rho^n \sin n\theta. \tag{17}$$

Using $f(\theta)$ to denote the function defined by (5), we then have to satisfy the condition

$$f(\theta) = \sum_{n=1}^{\infty} b_n 10^n \sin n\theta, \qquad 0 \leqq \theta \leqq \pi. \tag{18}$$

By the method of Art. 59, we get

$$b_n 10^n = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{200}{\pi}\, \theta \sin n\theta\, d\theta + \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} \frac{200}{\pi}\, (\pi - \theta) \sin n\theta\, d\theta$$

$$= \frac{800}{n^2\pi^2} \sin \frac{n\pi}{2}. \tag{19}$$

Consequently

$$b_1 = \frac{800}{\pi^2} \cdot \frac{1}{10}, \quad b_2 = 0, \quad b_3 = -\frac{800}{\pi^2} \cdot \frac{1}{10^3} \cdot \frac{1}{3^2}, \quad b_4 = 0, \cdots,$$

and insertion in (17) yields the required solution,

$$\tau(\rho, \theta) = \frac{800}{\pi^2}\left[\left(\frac{\rho}{10}\right)\sin\theta - \frac{1}{3^2}\left(\frac{\rho}{10}\right)^3\sin 3\theta + \frac{1}{5^2}\left(\frac{\rho}{10}\right)^5\sin 5\theta - \cdots\right]. \quad (20)$$

This series converges for $0 < \rho < 10$ and $0 < \theta < \pi$, and may therefore be used to find the steady-state temperature at each interior point of the plate.

EXERCISES

1. Find the temperature at a number of points on the axis of symmetry of the plate in the above example, and hence plot a curve showing the variation of temperature τ with distance ρ along this axis. From your curve, determine the distance ρ at which the temperature is 50°.

2. If, in the above example, boundary condition (5) is replaced by

$$\tau(10, \theta) = \frac{400}{\pi^2}(\pi\theta - \theta^2), \qquad 0 \leqq \theta \leqq \pi,$$

other conditions remaining the same, find the temperature function.

3. On the line $\theta = \pi/4$ of the plate of Exercise 2, find the distance ρ from the center of the circle at which the temperature is half its maximum value on this line.

4. A rectangular plate with insulated surfaces is 10 cm. wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error (see Exercise 5). If the temperature along one short edge ($y = 0$) is given by $\tau(x, 0) = 4(10x - x^2)$ deg., $0 < x < 10$, while the two long edges ($x = 0$ and $x = 10$), as well as the other short edge, are kept at 0° C., find the steady-state temperature function $\tau(x, y)$.

5. Find the temperature at the point (5, 5) of the plate of Exercise 4. Also find the distance y along the midsection $x = 5$ at which the temperature is 0.1 per cent of its maximum value, and hence justify the assumption of infinite length.

6. If the temperature along the edge $y = 0$ of the plate of Exercise 4 is given by $\tau(x, 0) = 20x$, $0 < x \leqq 5$, $\tau(x, 0) = 20(10 - x)$, $5 < x < 10$, other conditions remaining the same, find the temperature function $\tau(x, y)$.

7. Find the temperature at the point (3, 3) of the plate of Exercise 6. Show also that, for large values of y, the temperature at a point on the midsection of this plate is approximately 78.5 per cent of the value at the corresponding point of the plate of Exercise 4.

8. A square plate is bounded by the lines $x = 0$, $y = 0$, $x = 20$, and $y = 20$. Its faces are insulated and the temperature along the upper horizontal edge is given by $\tau(x, 20) = 20x - x^2$, $0 < x < 20$, while the other three edges are kept at 0° C. Find the steady-state temperature function $\tau(x, y)$.

9. Find the temperature at the center of the plate of Exercise 8.

10. Consider a rectangular plate, a by b cm. Let the temperature distributions along the four edges be given by

$$\tau(0, y) = f_1(y), \quad \tau(a, y) = f_2(y), \quad \tau(x, 0) = f_3(x), \quad \tau(x, b) = f_4(x).$$

Show that if τ_1, τ_2, τ_3, and τ_4 are four solutions of Laplace's equation, respectively, satisfying the boundary conditions:

$$\begin{array}{llll} \tau_1(0, y) = f_1(y), & \tau_1(a, y) = 0, & \tau_1(x, 0) = 0, & \tau_1(x, b) = 0; \\ \tau_2(0, y) = 0, & \tau_2(a, y) = f_2(y), & \tau_2(x, 0) = 0, & \tau_2(x, b) = 0; \\ \tau_3(0, y) = 0, & \tau_3(a, y) = 0, & \tau_3(x, 0) = f_3(x), & \tau_3(x, b) = 0; \\ \tau_4(0, y) = 0, & \tau_4(a, y) = 0, & \tau_4(x, 0) = 0, & \tau_4(x, b) = f_4(x); \end{array}$$

then the steady-state temperature function $\tau(x, y)$ is

$$\tau(x, y) = \tau_1(x, y) + \tau_2(x, y) + \tau_3(x, y) + \tau_4(x, y).$$

Hence, for any given boundary conditions, $\tau(x, y)$ can be found by solving at most four problems each of which involves three zero boundary values.

74. Flow of electricity. In Art. 33 we considered the flow of electricity in a cable having constant resistance R (ohms/mile), inductance L (henries/mile), capacitance C (farads/mile), and leakance G (mhos/mile). We there derived the so-called telephone equations,

$$\frac{\partial E}{\partial x} = -RI - L\frac{\partial I}{\partial t}, \tag{1}$$

$$\frac{\partial I}{\partial x} = -GE - C\frac{\partial E}{\partial t}, \tag{2}$$

where x (miles) is the distance measured along the cable from the source, t (sec.) is time, $E(x, t)$ (volts) is the potential, and $I(x, t)$ (amp.) is the current. By elimination of I and E in turn from equations (1) and (2), we also found the second order linear partial differential equations for E and I, respectively:

$$\frac{\partial^2 E}{\partial^2 x} = RGE + (RC + LG)\frac{\partial E}{\partial t} + LC\frac{\partial^2 E}{\partial t^2}, \tag{3}$$

$$\frac{\partial^2 I}{\partial x^2} = RGI + (RC + LG)\frac{\partial I}{\partial t} + LC\frac{\partial^2 I}{\partial t^2}. \tag{4}$$

When the effects of inductance and leakance are negligible, equations (1)–(4) reduce to the telegraph equations (Exercise 2, Art. 33), of which the last two are similar in form to the equation for one-dimensional heat flow. When the effects of resistance and leakance are negligible, the telephone equations reduce to the radio equations (Exercise 3, Art. 33); the last two of these are of the same form as the equation of the vibrating string.

However, when all four circuit parameters must be taken into account, the telephone equations are usually not similar in form to equations previously discussed. We shall deal here with an example illustrating this more general situation, but for simplicity the data have been chosen to make the computations relatively simple.

Example. A transmission line, for which $R = 1$ ohm/mile, $L = 0.01$ henry/mile, $C = 10^{-8}$ farad/mile, and $G = 10^{-6}$ mho/mile, is 1000 miles long. Initially the potential at each point is independent of time, with the ends $x = 0$ and $x = 1000$ at potentials $100e = 272$ and 100 volts, respectively. If the ends are suddenly grounded, find the resulting potential function $E(x, t)$.

Solution. Substituting the given values of R, L, C, and G in equation (3), we have

$$\frac{\partial^2 E}{\partial x^2} = 10^{-6}E + 2\cdot 10^{-8}\frac{\partial E}{\partial t} + 10^{-10}\frac{\partial^2 E}{\partial t^2}. \tag{5}$$

Assuming a solution of the form $E(x, t) = X(x)\cdot T(t)$, we get, by the usual process,

$$10^{10}\frac{X''}{X} = \frac{10000\,T + 200\,T' + T''}{T} = k, \tag{6}$$

a constant. This leads to three types of solutions,

$$E = (c_1 e^{10^{-5}\sqrt{k}x} + c_2 e^{-10^{-5}\sqrt{k}x})e^{-100t}(c_3 e^{\sqrt{k}t} + c_4 e^{-\sqrt{k}t}), \quad k > 0, \tag{7}$$

$$E = (c_5 \sin 10^{-5}\sqrt{-k}x + c_6 \cos 10^{-5}\sqrt{-k}x)e^{-100t}(c_7 \sin \sqrt{-k}t + c_8 \cos \sqrt{-k}t), \quad k < 0, \tag{8}$$

$$E = (c_9 x + c_{10})e^{-100t}(c_{11}t + c_{12}), \qquad k = 0. \tag{9}$$

Now grounding both ends of the line at time $t = 0$ gives us the boundary conditions

$$E(0, t) = 0, \qquad t > 0, \tag{10}$$

$$E(1000, t) = 0, \qquad t > 0. \tag{11}$$

It is easily seen that these relations can be met only by a solution of type (8). From (10) we then get $c_6 = 0$; and from (11), $\sin 10^{-2}\sqrt{-k} = 0$, whence $\sqrt{-k} = 100\,n\pi$, n being an integer. We therefore have to deal with

$$E = c_5 \sin\frac{n\pi x}{1000}\cdot e^{-100t}(c_7 \sin 100n\pi t + c_8 \cos 100n\pi t), \tag{12}$$

which satisfies the differential equation (5) and boundary conditions (10)–(11).

Before expressing and applying the initial condition furnished by the statement of the problem, we further restrict (12) as follows. Since the potential at any point of the line before the change is independent of time, and because of the inertial effect of inductance, the initial time rate of change of potential will be zero:

$$\left.\frac{\partial E}{\partial t}\right]_{t=0} = 0, \qquad 0 < x < 1000. \tag{13}$$

From (12) we find

$$\frac{\partial E}{\partial t} = c_5 \sin \frac{n\pi x}{1000} \cdot e^{-100t}(100n\pi c_7 \cos 100n\pi t - 100n\pi c_8 \sin 100n\pi t$$
$$- 100c_7 \sin 100n\pi t - 100c_8 \cos 100n\pi t), \tag{14}$$

and (13) thus yields $100n\pi c_7 - 100c_8 = 0$, or $c_8 = n\pi c_7$. Consequently (12) becomes

$$E = c_5c_7 \sin \frac{n\pi x}{1000} \cdot e^{-100t} (\sin 100n\pi t + n\pi \cos 100n\pi t). \tag{15}$$

To formulate the remaining initial condition, we need a solution of (5) that is independent of t. When E depends only upon x, $E = E_s(x)$, (5) reduces to

$$\frac{d^2E_s}{dx^2} = 10^{-6}E_s, \tag{16}$$

which has the solution

$$E_s(x) = ae^{x/1000} + be^{-x/1000}. \tag{17}$$

Since $E_s(0) = 100e$ and $E_s(1000) = 100$, we get from (17),

$$100e = a + b, \qquad 100 = ae + be^{-1},$$

whence $a = 0$, $b = 100e = 272$. Consequently our other initial condition is

$$E(x, 0) = 272e^{-x/1000}, \qquad 0 < x < 1000. \tag{18}$$

It is apparent that (18) can be fulfilled only by taking an infinite series of terms of type (15),

$$E = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{1000} \cdot e^{-100t} (\sin 100n\pi t + n\pi \cos 100n\pi t). \tag{19}$$

Applying (18), we find

$$272e^{-x/1000} = \sum_{n=1}^{\infty} (n\pi b_n) \sin \frac{n\pi x}{1000}. \tag{20}$$

Accordingly, we must have (Art. 59)

$$n\pi b_n = \frac{2}{1000} \int_0^{1000} 272e^{-x/1000} \sin \frac{n\pi x}{1000}\, dx,$$

which yields

$$b_n = \frac{544(1 - e^{-1}\cos n\pi)}{1 + n^2\pi^2}. \tag{21}$$

Inserting these values in (19), we therefore get

$$E = 544\sum_{n=1}^{\infty}\frac{1 - e^{-1}\cos n\pi}{1 + n^2\pi^2}\sin\frac{n\pi x}{1000}\cdot e^{-100t}(\sin 100n\pi t + n\pi\cos 100n\pi t). \tag{22}$$

This is the desired potential function.

EXERCISES

1. Using the potential function (22), find $\partial I/\partial x$ from equation (2). Integrate this partially with respect to x, and determine the arbitrary function of t thereby introduced by means of equation (1) together with the condition

$$\left.\frac{\partial I}{\partial t}\right]_{t=0} = 0.$$

Hence show that the current function $I(x, t)$ is given by

$$I = 0.544\sum_{n=1}^{\infty}\frac{1 - e^{-1}\cos n\pi}{1 + n^2\pi^2}\cos\frac{n\pi x}{1000}\cdot e^{-100t}(\cos 100n\pi t - n\pi\sin 100n\pi t)$$
$$+ 0.172e^{-100t}.$$

2. Using the result of Exercise 1, find the current at the midpoint of the line when $t = 0.01$ sec.

3. Show that, in general, equation (3) has five distinct types of solutions, according as

$$(1)\quad k < -\frac{(RC - LG)^2}{4LC}; \qquad (2)\quad k = -\frac{(RC - LG)^2}{4LC};$$

$$(3)\quad -\frac{(RC - LG)^2}{4LC} < k < 0; \qquad (4)\quad k = 0; \qquad (5)\quad k > 0.$$

Why were only three types obtained in the example?

4. A transmission line is 100 miles long, and $R = 10$ ohms/mile, $C = 10^{-6}$ farad/mile, and L and G are negligible. Initially it is under steady-state conditions, with potential 1100 volts at the source ($x = 0$) and 1000 volts at the load. The terminal end is suddenly grounded, reducing its potential to zero, but the potential at the source is kept at 1100 volts. Find the potential function $E(x, t)$.

5. Find the current at each end of the line of Exercise 4, 0.01 sec. after the change.

6. If both ends of the line of Exercise 4 are initially grounded, the line being uncharged, and a potential of 1000 volts is suddenly applied at the source, find the potential function $E(x, t)$.

7. Find the percentage of the maximum current value attained at the gounded end of the line of Exercise 6 at the end of 0.01, 0.02, and 0.05 sec.

8. A line is 10 miles long, and $L = 1$ henry/mile, $C = 10^{-6}$ farad/mile, and R and G are negligible. Initially it is under steady-state conditions, with a potential of 100 volts at each point. If the terminal end is suddenly grounded, find the resulting potential function $E(x, t)$.

9. Using the fact that $I(x, 0) = 0$ for the line of Exercise 8, find the current function $I(x, t)$, and hence determine the current at the midpoint of the line when $t = 0.025$ sec.

10. The line of the example is an instance of a distortionless line, for which $RC = LG$. Show that the substitution $E = E_1e^{-100t}$ (cf. Exercise 5, Art. 33) transforms equation (5) into the form of one of the radio equations, and hence solve the problem anew by the method used in Exercise 8 above.

CHAPTER VIII

NON-LINEAR EQUATIONS OF SECOND ORDER

As was stated at the beginning of Chapter VII, partial differential equations of order higher than the first present complications which make the systematic classification of such equations and their solutions difficult. This is particularly true of non-linear equations of higher orders, and we shall therefore discuss, in this final chapter, only a few types of non-linear equations of second order.

The methods of solving second order non-linear equations to be considered here are, in several cases, applicable also to second order linear equations. We shall sometimes use a linear equation to illustrate, in a simple manner, the process under discussion.

75. Intermediate integrals. One way of attacking the problem of solving a given second order differential equation is to seek a solvable first order equation which is such that the original second order relation can be obtained from it and the two relations derived by partial differentiation. Such a first order equation is called an *intermediate integral*.

We have already dealt with intermediate integrals, although they were not so designated. Thus, in Example 3 of Art. 29, elimination of the arbitrary function from the relation $q = f(p)$ led to the second order equation $rt - s^2 = 0$, and consequently the first order equation, $q = f(p)$, is an intermediate integral of $rt - s^2 = 0$. Likewise, integration with respect to x of the equation $xr - ys + p = y^2$ (Example 4, Art. 67) gave us the intermediate integral $xp - yq = xy^2 + \phi(y)$.

An intermediate integral may contain no arbitrary elements, or it may contain arbitrary constants, or, as in each of the above examples, it may contain an arbitrary function. We cannot expect more than one arbitrary function to appear in an intermediate integral, for the elimination of two or more arbitrary functions from a first order relation will, in general, lead to a differential equation of order higher than the second.

We therefore consider the most general situation, in which the intermediate integral is of the form

$$\phi(u, v) = 0, \tag{1}$$

where u and v are definite expressions involving x, y, z, p, and q, and ϕ is an arbitrary function. By elimination of ϕ from (1), we shall find the form possessed by the second order partial differential equation having (1) as intermediate integral. Differentiation of (1) with respect to x and y in turn yields the relations (Art. 18)

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p + \frac{\partial u}{\partial p} r + \frac{\partial u}{\partial q} s\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p + \frac{\partial v}{\partial p} r + \frac{\partial v}{\partial q} s\right) = 0, \quad (2)$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q + \frac{\partial u}{\partial p} s + \frac{\partial u}{\partial q} t\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q + \frac{\partial v}{\partial p} s + \frac{\partial v}{\partial q} t\right) = 0. \quad (3)$$

Equating the values of ϕ_u/ϕ_v given by (2) and (3), and reducing, we get the equation

$$R_1 r + S_1 s + T_1 t + U_1(rt - s^2) = F_1, \quad (4)$$

where

$$R_1 = \frac{\partial(u, v)}{\partial(p, y)} + \frac{\partial(u, v)}{\partial(p, z)} q, \qquad T_1 = \frac{\partial(u, v)}{\partial(x, q)} + \frac{\partial(u, v)}{\partial(z, q)} p,$$

$$S_1 = \frac{\partial(u, v)}{\partial(x, p)} + \frac{\partial(u, v)}{\partial(z, p)} p + \frac{\partial(u, v)}{\partial(q, y)} + \frac{\partial(u, v)}{\partial(q, z)} q, \quad (5)$$

$$U_1 = \frac{\partial(u, v)}{\partial(p, q)}, \qquad F_1 = \frac{\partial(u, v)}{\partial(y, x)} + \frac{\partial(u, v)}{\partial(y, z)} p + \frac{\partial(u, v)}{\partial(z, x)} q;$$

the symbol $\partial(u, v)/\partial(\alpha, \beta)$ denotes, as usual, a Jacobian (Art. 24):

$$\frac{\partial(u, v)}{\partial(\alpha, \beta)} = \frac{\partial u}{\partial \alpha}\frac{\partial v}{\partial \beta} - \frac{\partial u}{\partial \beta}\frac{\partial v}{\partial \alpha}.$$

From this elimination process we deduce the result that *any second order partial differential equation which possesses an intermediate integral of the form* (1) *must itself be of the form* (4). Accordingly, if a second order equation is not of the form (4), it would be useless to try to find an intermediate integral of type (1).

Moreover, it should be noticed that the converse of the above statement is not necessarily true. That is, when an equation has the form (4), it does not follow that it must have an intermediate integral of type (1).

From relations (5) we see in particular that when either u or v lacks both p and q, and in certain other cases, $U_1 = 0$, and equation (4) is of the first degree in the second derivatives r, s, and t. For convenience, we shall call an equation (4), in which $U_1 = 0$, a *uniform* equation; and if $U_1 \neq 0$, we shall say that (4) is a *non-uniform* equation. In the two following articles, we shall discuss Monge's methods of seeking intermediate integrals of these two types of equations.

76. Monge's method: uniform equations. Consider the uniform equation

$$Rr + Ss + Tt = F, \tag{1}$$

where R, S, T, and F are functions of x, y, z, p, and q. Whether or not this equation possesses an intermediate integral of the form

$$\phi(u, v) = 0, \tag{2}$$

we have at present no means of telling. However, assuming that (2) is an intermediate integral of (1), we can conduct our search for the expressions u and v as follows.

From the differential relations

$$dp = \frac{\partial p}{\partial x}\,dx + \frac{\partial p}{\partial y}\,dy = r\,dx + s\,dy,$$

$$dq = \frac{\partial q}{\partial x}\,dx + \frac{\partial q}{\partial y}\,dy = s\,dx + t\,dy,$$

we get

$$r = \frac{dp - s\,dy}{dx}, \qquad t = \frac{dq - s\,dx}{dy}.$$

Substituting these expressions for r and t in (1), we have

$$R\,\frac{dp - s\,dy}{dx} + Ss + T\,\frac{dq - s\,dx}{dy} = F,$$

or

$$R\,dy\,dp + T\,dx\,dq - F\,dx\,dy = s[R(dy)^2 - S\,dx\,dy + T(dx)^2]. \tag{3}$$

Evidently any functional relations simultaneously satisfying

$$R\,dy\,dp + T\,dx\,dq - F\,dx\,dy = 0, \tag{4}$$

$$R(dy)^2 - S\,dx\,dy + T(dx)^2 = 0, \tag{5}$$

will satisfy (3), and consequently the given equation (1).

Now let

$$f(x, y, z, p, q) = 0 \tag{6}$$

be an intermediate integral of (1), formed by choosing the ϕ-function (2) in some way; in particular, f may be either u or v. We shall deduce the fact that the differential relations (4) and (5), known as *Monge's equations*, may be used to determine an intermediate integral when one exists.

Differentiating (6) with respect to x and y in turn, we get

$$f_x + f_z p + f_p r + f_q s = 0, \qquad f_y + f_z q + f_p s + f_q t = 0, \tag{7}$$

whence

$$r = -\frac{f_x + f_z p + f_q s}{f_p}, \qquad t = -\frac{f_y + f_z q + f_p s}{f_q}.$$

Inserting these in (1), we find

$$-R\frac{f_x + f_z p + f_q s}{f_p} + Ss - T\frac{f_y + f_z q + f_p s}{f_q} = F,$$

or

$$Rf_x f_q + Rf_z f_q p + Tf_y f_p + Tf_z f_p q + Ff_p f_q + s(Rf_q^2 - Sf_p f_q + Tf_p^2) = 0. \tag{8}$$

Now to say that (6) is an intermediate integral of (1) means (Art. 75) that (1) is satisfied by virtue of (6) and the two derived relations (7). Hence (8) is not an independent equation determining s; instead, we must have *

$$Rf_x f_q + Rf_z f_q p + Tf_y f_p + Tf_z f_p q + Ff_p f_q = 0, \tag{9}$$

$$Rf_q^2 - Sf_p f_q + Tf_p^2 = 0. \tag{10}$$

These two equations to be satisfied by f contain five independent variables x, y, z, p, and q, and the five first partial derivatives of f. We may then use Jacobi's method (Art. 51) to find other functional relations of the type (6). The equations subsidiary to (9) are, in part,

$$\frac{dx}{-Rf_q} = \frac{dy}{-Tf_p} = \frac{dz}{-Rf_q p - Tf_p q} = \frac{dp}{-Tf_y - Tf_z q - Ff_q}$$

$$= \frac{dq}{-Rf_x - Rf_z p - Ff_p}.$$

Denoting the common value of these ratios by $d\lambda$, we get for the differential expression in (4),

$$R\,dy\,dp + T\,dx\,dq - F\,dx\,dy = (RT^2 f_y f_p + RT^2 f_z f_p q + FRTf_p f_q$$

$$+ R^2 Tf_x f_q + R^2 Tf_z f_q p + FRTf_p f_q - FRTf_p f_q)(d\lambda)^2$$

$$= RT(Rf_x f_q + Rf_z f_q p + Tf_y f_p + Tf_z f_p q + Ff_p f_q)(d\lambda)^2,$$

* We put aside the trivial case in which s, rather than its coefficient (10), vanishes. For, unless (1) is itself the simple linear equation $s = 0$, with the solution $z = \phi(x) + \psi(y)$, (1) will have another type of solution which we wish to find by Monge's method.

which vanishes by (9) itself. Similarly, we get for the left member of (5),

$$R(dy)^2 - S\,dx\,dy + T(dx)^2 = (RT^2f_p^2 - RSTf_pf_q + R^2Tf_q^2)(d\lambda)^2$$
$$= RT(Rf_q^2 - Sf_pf_q + Tf_p^2)(d\lambda)^2,$$

which also vanishes, by (10). Therefore (4) and (5) may be employed to find intermediate integrals.

Since, in any given case, we do not know if (1) possesses an intermediate integral, the above argument should be regarded as merely suggestive of a tentative way of seeking the desired relation (2).

We may deal with Monge's equations (4) and (5) in the following manner. The differential relation (5), which is essentially a quadratic equation in dy/dx, yields, in general, two linear differential relations,

$$dy = \alpha\,dx, \qquad dy = \beta\,dx, \tag{11}$$

where α and β, like R, S, T, and F, are functions of x, y, z, p, and q. These will be identical if (5) is a perfect square; this situation is dealt with below, and illustrated in Example 2. Another exceptional case arises when $R = 0$; we may then use, instead of (11), the two relations

$$dx = 0, \qquad dx = \beta_1\,dy. \tag{11'}$$

Combining relations (11), in turn, with equation (4), we get the two linear differential systems,

$$dy = \alpha\,dx, \qquad R\alpha\,dp + T\,dq - F\alpha\,dx = 0, \tag{12}$$
$$dy = \beta\,dx, \qquad R\beta\,dp + T\,dq - F\beta\,dx = 0. \tag{13}$$

With either of these pairs we may also use the relation

$$dz = p\,dx + q\,dy. \tag{14}$$

Suppose first that each system gives us a pair of integrals: $u_1 = a_1$ and $v_1 = b_1$ from (12), and $u_2 = a_2$ and $v_2 = b_2$ from (13), where the a's and b's are arbitrary constants. Then we deduce that

$$u_1 = \phi_1(v_1), \qquad u_2 = \phi_2(v_2), \tag{15}$$

where ϕ_1 and ϕ_2 are arbitrary, are two intermediate integrals. If we can solve the two equations (15) for p and q, and, after substitution in (14), if we can then integrate this relation, we have a solution of equation (1). Evidently there may be two arbitrary functions in the result.

If, however, only one intermediate integral, $u = \phi(v)$, is obtainable (which will, in particular, be the case when (5) is a perfect square), we can try to integrate this first order equation, using the methods of Chapters IV and V. Likewise, if two intermediate integrals have been found, but p and q cannot be determined from them, or if the values of p and q do not permit us to integrate (14), we try to integrate one of the intermediate integrals.

Here we shall carry the process, when it can be effected, only as far as the determination of the intermediate integral or integrals. In Art. 78, we shall investigate the remainder of the process.

Example 1. Apply Monge's method to the non-linear equation

$$q(1 + q)r - p(1 + 2q)s + p^2t = 0.$$

Solution. Equation (5) is here

$$q(1 + q)(dy)^2 + p(1 + 2q)dx\,dy + p^2(dx)^2 = 0,$$

or

$$[(1 + q)\,dy + p\,dx][q\,dy + p\,dx] = 0.$$

Consequently we have

$$dy = -\frac{p}{1 + q}\,dx, \qquad dy = -\frac{p}{q}\,dx,$$

so that

$$\alpha = -\frac{p}{1 + q}, \qquad \beta = -\frac{p}{q}.$$

Equations (12) then are

$$dy + q\,dy + p\,dx = 0, \qquad -pq\,dp + p^2\,dq = 0.$$

With the aid of (14), the first of these becomes $dy + dz = 0$, whence

$$y + z = a_1.$$

The second relation gives us, in addition,

$$\frac{dq}{q} - \frac{dp}{p} = 0,$$

$$\log q - \log p = \log b_1,$$

$$\frac{q}{p} = b_1.$$

We therefore have

$$q = p\phi_1(y + z),$$

which is easily shown to be an intermediate integral.

Likewise, we find from relations (13), together with (14),

$$p\,dx + q\,dy = dz = 0, \qquad z = a_2;$$

$$-p(1 + q)\,dp + p^2\,dq = 0,$$

$$\frac{p\,dq - (1 + q)\,dp}{p^2} = 0, \qquad \frac{1 + q}{p} = b_2;$$

and consequently there is obtained

$$1 + q = p\phi_2(z),$$

which also may be verified as an intermediate integral.

Thus the given equation in this case possesses two intermediate integrals of the desired form. We shall deal with these further in Art. 78.

Example 2. Apply Monge's method to the non-linear equation

$$xq^2r - 2xpqs + xp^2t = 2pq^2.$$

Solution. From (5) we get

$$xq^2(dy)^2 + 2xpq\,dx\,dy + xp^2(dx)^2 = 0,$$

or

$$(q\,dy + p\,dx)^2 = 0.$$

We therefore have the double root

$$\alpha = \beta = -\frac{p}{q},$$

and consequently can obtain only one intermediate integral at best. Monge's equations then yield

$$p\,dx + q\,dy = dz = 0, \qquad z = a;$$

$$-xpq\,dp + xp^2dq - 2p^2q\,dx = 0,$$

$$\frac{dp}{p} - \frac{dq}{q} + 2\frac{dx}{x} = 0, \qquad \frac{x^2p}{q} = b.$$

Hence we have

$$x^2p = q\phi(z),$$

which may be shown to be an intermediate integral.

EXERCISES

Using Monge's method, find, wherever possible, intermediate integrals of the following equations.

1. $qr - ps = 0$.
2. $qr - (p + q)s + pt = 0$.
3. $q^2r - 2pqs + p^2t = 0$.
4. $xr - ys + p = y^2$.
5. $r + s = 4xy + 2y^2$.
6. $xs - 2x^2t = 1$.
7. $xyr + x^2s = yp + (x^2 - y)e^y$.
8. $xs + yt = 12xy^2 - q$.
9. $qr - (1 + p + q)s + (1 + p)t = 0$.
10. $x^2r + xys = z - xp - yq$.
11. $2yr + s = 2ye^y + z + (1 - 2y)p - q$.
12. $x(1 + q)r + x(1 - p + q)s - xpt = (1 + p + q)(1 + q)$.
13. $xq^2r - q(z + 2xp)s + p(xp + z)t + pq^2 = 0$.
14. $xq^2r - 2xpqs + xp^2t = xp^2q - 2pq^2$.
15. $q(1 - q)r - (1 + p - q - 2pq)s - p(1 + p)t = 0$.

77. Monge's method: non-uniform equations. Consider next the non-uniform equation

$$Rr + Ss + Tt + U(rt - s^2) = F, \tag{1}$$

where R, S, T, U and F are functions of x, y, z, p, and q, and where $U \neq 0$. By substituting

$$r = \frac{dp - s\,dy}{dx}, \qquad t = \frac{dq - s\,dx}{dy},$$

as in the uniform equation of Art. 76, we get

$$R\,dy\,dp + T\,dx\,dq + U\,dp\,dq - F\,dx\,dy$$
$$= s[R(dy)^2 - S\,dx\,dy + T(dx)^2 + U(dx\,dp + dy\,dq)]. \tag{2}$$

By analogy with the procedure of Art. 76, we deal with the differential relations

$$R\,dy\,dp + T\,dx\,dq + U\,dp\,dq - F\,dx\,dy = 0, \tag{3}$$

$$R(dy)^2 - S\,dx\,dy + T(dx)^2 + U(dx\,dp + dy\,dq) = 0; \tag{4}$$

for, any functional relation simultaneously satisfying (3) and (4) will necessarily satisfy (2), and therefore (1). Using the results of Art. 75, we shall show that if $u = a$ and $v = b$ are two integrals of (3) and (4), such that $\partial(u, v)/\partial(p, q) \equiv u_pv_q - u_qv_p \neq 0$, then

$$\phi(u, v) = 0, \tag{5}$$

where ϕ is arbitrary, is an intermediate integral of (1).

From $u = a$ and $v = b$ we find (Art. 19)

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz + \frac{\partial u}{\partial p}dp + \frac{\partial u}{\partial q}dq = 0,$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz + \frac{\partial v}{\partial p}dp + \frac{\partial v}{\partial q}dq = 0.$$

Inserting

$$dz = p\,dx + q\,dy, \tag{6}$$

these equations become

$$(u_x + pu_z)\,dx + (u_y + qu_z)\,dy + u_p\,dp + u_q\,dq = 0, \tag{7}$$

$$(v_x + pv_z)\,dx + (v_y + qv_z)\,dy + v_p\,dp + v_q\,dq = 0. \tag{8}$$

Using the notation (cf. Art. 75)

$$R_1 = \frac{\partial(u, v)}{\partial(p, y)} + \frac{\partial(u, v)}{\partial(p, z)}q, \qquad T_1 = \frac{\partial(u, v)}{\partial(x, q)} + \frac{\partial(u, v)}{\partial(z, q)}p,$$

$$S_1 = \frac{\partial(u, v)}{\partial(x, p)} + \frac{\partial(u, v)}{\partial(z, p)}p + \frac{\partial(u, v)}{\partial(q, y)} + \frac{\partial(u, v)}{\partial(q, z)}q, \tag{9}$$

$$U_1 = \frac{\partial(u, v)}{\partial(p, q)}, \qquad F_1 = \frac{\partial(u, v)}{\partial(y, x)} + \frac{\partial(u, v)}{\partial(y, z)}p + \frac{\partial(u, v)}{\partial(z, x)}q,$$

and remembering that, by hypothesis, $U_1 \neq 0$, we may solve (7) and (8) for dp and dq; we get

$$U_1\,dp = -T_1\,dx + \left[\frac{\partial(u, v)}{\partial(q, y)} + \frac{\partial(u, v)}{\partial(q, z)}q\right]dy, \tag{10}$$

$$U_1\,dq = -R_1\,dy + \left[\frac{\partial(u, v)}{\partial(x, p)} + \frac{\partial(u, v)}{\partial(z, p)}p\right]dx. \tag{11}$$

Multiplying the first of these equations by dx, and the second by dy, and adding, we find

$$U_1(dx\,dp + dy\,dq) = -T_1(dx)^2 - R_1(dy)^2 + S_1dx\,dy,$$

or

$$R_1(dy)^2 - S_1dx\,dy + T_1(dx)^2 + U_1(dx\,dp + dy\,dq) = 0. \tag{12}$$

Similarly, equations (10) and (11) lead to

$$(U_1\,dp + T_1\,dx)(U_1\,dq + R_1\,dy) = (F_1U_1 + R_1T_1)\,dx\,dy,$$

whence

$$R_1\,dy\,dp + T_1\,dx\,dq + U_1\,dp\,dq - F_1\,dx\,dy = 0. \tag{13}$$

Comparison of equations (12) and (13) with relations (3) and (4) shows that, since $u = a$ and $v = b$ satisfy each pair, we must have

$$\frac{R_1}{R} = \frac{S_1}{S} = \frac{T_1}{T} = \frac{U_1}{U} = \frac{F_1}{F}.$$

Hence equation (1) is equivalent to

$$R_1 r + S_1 s + T_1 t + U_1(rt - s^2) = F_1. \tag{14}$$

But this equation was found (Art. 75) to be the eliminant corresponding to (5). Therefore, *if* $u = a$ *and* $v = b$, *with* $\partial(u, v)/\partial(p, q) \neq 0$, *satisfy Monge's equations* (3) *and* (4), *relation* (5) *will be an intermediate integral of the given equation* (1).

To find the needed integrals $u = a$ and $v = b$ of Monge's equations (3) and (4), we proceed as follows. Because of the occurrence in (4) of the term $U(dx\,dp + dy\,dq)$, we cannot now factor the left member of (4), as we could the corresponding expression in Art. 76, nor can we factor the left member of (3). We therefore form the linear combination

$$R(dy)^2 - S\,dx\,dy + T(dx)^2 + U(dx\,dp + dy\,dq)$$
$$+ \lambda(R\,dy\,dp + T\,dx\,dq + U\,dp\,dq - F\,dx\,dy) = 0,$$

or, denoting the left member of this equation by M,

$$M \equiv R(dy)^2 - (S + \lambda F)\,dx\,dy + T(dx)^2 + U\,dx\,dp + U\,dy\,dq$$
$$+ \lambda R\,dy\,dp + \lambda T\,dx\,dq + \lambda U\,dp\,dq = 0, \tag{15}$$

and try to determine λ so that M is factorable. Since M contains no terms involving $(dp)^2$ or $(dq)^2$, dp can appear in only one factor and dq only in the other. The factors may consequently be assumed to have the forms

$$A_1\,dy + B_1\,dx + C_1\,dp, \qquad A_2\,dy + B_2\,dx + C_2\,dq.$$

Comparing the product of these with M, we see that we must have

$$A_1A_2 = R, \tag{16}$$

$$A_1B_2 + B_1A_2 = -(S + \lambda F), \tag{17}$$

$$B_1B_2 = T, \tag{18}$$

$$C_1B_2 = U, \tag{19}$$

$$A_1C_2 = U, \tag{20}$$

$$C_1A_2 = \lambda R, \tag{21}$$

$$B_1C_2 = \lambda T, \tag{22}$$

$$C_1C_2 = \lambda U. \tag{23}$$

If we take

$$A_1 = 1, \qquad A_2 = R,$$

(16) is satisfied, and (20) and (21) respectively yield

$$C_2 = U, \qquad C_1 = \lambda.$$

These values evidently satisfy (23), and then (19) and (22) give us

$$B_2 = \frac{U}{\lambda}, \qquad B_1 = \frac{\lambda T}{U}.$$

These also satisfy (18). Substituting in the remaining relation (17), we get as the equation to be satisfied by λ,

$$\frac{U}{\lambda} + \frac{\lambda RT}{U} = -S - \lambda F,$$

or

$$(RT + FU)\lambda^2 + SU\lambda + U^2 = 0. \tag{24}$$

The quadratic equation (24) will, in general, yield two values of λ; call them α and β. For each of these in turn, M has linear differential factors:

$$M \equiv \left(dy + \frac{\alpha T}{U}\,dx + \alpha\,dp\right)\left(R\,dy + \frac{U}{\alpha}\,dx + U\,dq\right) = 0,$$

$$M \equiv \left(dy + \frac{\beta T}{U}\,dx + \beta\,dp\right)\left(R\,dy + \frac{U}{\beta}\,dx + U\,dq\right) = 0,$$

whence

$$(U\,dy + \alpha T\,dx + \alpha U\,dp)(\alpha R\,dy + U\,dx + \alpha U\,dq) = 0, \tag{25}$$

$$(U\,dy + \beta T\,dx + \beta U\,dp)(\beta R\,dy + U\,dx + \beta U\,dq) = 0. \tag{26}$$

To determine the functions u and v, we must combine one factor of (25) with one of (26). Now the first factors taken together,

$$U\,dy + \alpha T\,dx + \alpha U\,dp = 0, \qquad U\,dy + \beta T\,dx + \beta U\,dp = 0,$$

yield, upon subtraction,

$$(\alpha - \beta)(T\,dx + U\,dp) = 0;$$

if $\alpha \neq \beta$, this implies $T\,dx + U\,dp = 0$, and consequently $U\,dy = 0$. This cannot furnish a solution. Likewise, the second factors in (25) and (26) together lead to $U\,dx = 0$, which is of no value to us. We must therefore take as the two relevant systems

$$\begin{aligned} U\,dy + \alpha T\,dx + \alpha U\,dp &= 0, \\ \beta R\,dy + U\,dx + \beta U\,dq &= 0; \end{aligned} \tag{27}$$

and

$$\begin{aligned} U\,dy + \beta T\,dx + \beta U\,dp &= 0, \\ \alpha R\,dy + U\,dx + \alpha U\,dq &= 0. \end{aligned} \tag{28}$$

When $\alpha = \beta$, which occurs when (24) is a perfect square, the systems (27) and (28) will, of course, coalesce into one pair of equations. In this case we cannot hope for more than one intermediate integral. But if the quadratic equation (24) yields two distinct values of λ, we have two possible systems of differential relations; solutions $u = a$ and $v = b$ of either pair, when obtainable, give us an intermediate integral, $\phi(u, v) = 0$.

If, in a given problem, $RT + FU = 0$, equation (24) will not be a quadratic equation. Setting $\lambda = 1/\mu$, (24) takes the form

$$U^2\mu^2 + SU\mu + RT + FU = 0, \tag{24'}$$

which has a zero root when $RT + FU = 0$. In this case, therefore, we deal with the systems

$$\begin{aligned} T\,dx + U\,dp &= 0, \\ \beta R\,dy + U\,dx + \beta U\,dq &= 0, \end{aligned} \tag{27'}$$

and

$$\begin{aligned} U\,dy + \beta T\,dx + \beta U\,dp &= 0, \\ R\,dy + U\,dq &= 0, \end{aligned} \tag{28'}$$

obtained by dividing the first and last relations of (27) and (28) by α and then allowing α to become infinite.

Example 1. Apply Monge's method to the equation

$$r - 3s - 4t + 2(rt - s^2) = 3.$$

Solution. Here equation (24) is

$$2\lambda^2 - 6\lambda + 4 = 0,$$

or

$$\lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0,$$

whence we get two distinct roots,

$$\alpha = 1, \qquad \beta = 2.$$

The system (27) then becomes

$$2\,dy - 4\,dx + 2\,dp = 0, \qquad 2\,dy + 2\,dx + 4\,dq = 0.$$

These are immediately integrable; we get

$$u_1 \equiv y - 2x + p = a_1, \qquad v_1 \equiv y + x + 2q = b_1.$$

Therefore

$$y - 2x + p = \phi_1(y + x + 2q),$$

where ϕ_1 is arbitrary, is an intermediate integral. From relations (28) we also get

$$2\,dy - 8\,dx + 4\,dp = 0, \qquad dy + 2\,dx + 2\,dq = 0,$$

whence

$$u_2 \equiv y - 4x + 2p = a_2, \qquad v_2 \equiv y + 2x + 2q = b_2.$$

Hence we obtain a second intermediate integral,

$$y - 4x + 2p = \phi_2(y + 2x + 2q),$$

where ϕ_2 is arbitrary.

Example 2. Apply Monge's method to the equation

$$x^2r + (q - xy)s - 2pt - x(rt - s^2) = yq - 2xp.$$

Solution. Equation (24) gives us

$$-xyq\lambda^2 - x(q - xy)\lambda + x^2 = 0,$$

or

$$yq\lambda^2 + (q - xy)\lambda - x = (q\lambda - x)(y\lambda + 1) = 0.$$

Consequently

$$\alpha = \frac{x}{q}, \qquad \beta = -\frac{1}{y}.$$

From (27) we get

$$-x\,dy - \frac{2xp}{q}\,dx - \frac{x^2}{q}\,dp = 0, \qquad -\frac{x^2}{y}\,dy - x\,dx + \frac{x}{y}\,dq = 0,$$

or

$$q\,dy + 2p\,dx + x\,dp = 0, \qquad x\,dy + y\,dx - dq = 0.$$

The first of these may be written as

$$x\,dp + p\,dx + p\,dx + q\,dy = x\,dp + p\,dx + dz = 0,$$

so that $u_1 \equiv xp + z = a_1$. From the other we get directly, $v_1 \equiv xy - q = b_1$. Hence

$$xp + z = \phi(xy - q)$$

is an intermediate integral. Relations (28) give us

$$-x\,dy + \frac{2p}{y}\,dx + \frac{x}{y}\,dp = 0, \qquad \frac{x^3}{q}\,dy - x\,dx - \frac{x^2}{q}\,dq = 0,$$

or

$$xy\,dy - 2p\,dx - x\,dp = 0, \qquad x^2\,dy - q\,dx - x\,dq = 0.$$

However, neither of these differential expressions, nor any derivable from them, is integrable, as may be proved by the usual test (Art. 19). Consequently Monge's method furnishes us with only one intermediate integral in this case.

EXERCISES

Using Monge's method, find, wherever possible, intermediate integrals of the following equations.

1. $7s - 2(rt - s^2) = -3$.
2. $s + s^2 - rt = 2$.
3. $r + 2t + s^2 - rt = 3$.
4. $r - 2s + t + rt - s^2 = 0$.
5. $2r + 2s + 2t - rt + s^2 = 3$.
6. $2r - s - t - rt + s^2 + 2 = 0$.
7. $r - 2t + rt - s^2 = 2$.
8. $2(x - y)s + rt - s^2 + 4xy = 0$.
9. $yr - ps + t + y(rt - s^2) + 1 = 0$.
10. $xqr + (p - q)s + ypt + (1 + xy)(rt - s^2) + pq = 0$.

78. Solutions from intermediate integrals. In this article we shall consider a few ways of solving second order partial differential equations for which intermediate integrals have been found by Monge's methods. We shall illustrate these processes by further discussion of the equations used as examples in Arts. 76 and 77.

Example 1. Solve the equation

$$q(1 + q)r - p(1 + 2q)s + p^2t = 0 \tag{1}$$

(Example 1, Art. 76).

Solution. In Art. 76 we found two intermediate integrals of this equation,

$$q = p\phi_1(y + z), \qquad 1 + q = p\phi_2(z). \tag{2}$$

These may be solved for p and q; we get

$$p = \frac{1}{\phi_2(z) - \phi_1(y + z)}, \qquad q = \frac{\phi_1(y + z)}{\phi_2(z) - \phi_1(y + z)}.$$

Inserting these expressions in $dz = p\,dx + q\,dy$, and clearing of fractions, we have

$$\phi_2(z)\,dz - \phi_1(y + z)\,dz = dx + \phi_1(y + z)\,dy,$$

or

$$\phi_2(z)\,dz = dx + \phi_1(y+z)\,(dy+dz). \tag{3}$$

This is immediately integrable, and we find as a solution of the given equation,

$$f(z) = x + g(y+z), \tag{4}$$

where f and g are new arbitrary functions.

Example 2. Solve the equation

$$xq^2r - 2xpqs + xp^2t = -2pq^2 \tag{5}$$

(Example 2, Art. 76).

Solution. Only one intermediate integral of this equation was obtainable by Monge's method, namely,

$$x^2p - q\phi(z) = 0. \tag{6}$$

We may treat this relation by Lagrange's method; the subsidiary equations are

$$\frac{dx}{x^2} = \frac{dy}{-\phi(z)}, \qquad dz = 0.$$

From the second of these we get $z = a$, a constant. Using this in the first equation, we get

$$dy + \frac{\phi(a)}{x^2}\,dx = 0,$$

$$y - \frac{\phi(a)}{x} = b,$$

whence the given equation is found to have the solution

$$xy = \phi(z) + x\psi(z), \tag{7}$$

containing the two arbitrary functions ϕ and ψ.

Example 3. Solve the equation

$$r - 3s - 4t + 2(rt - s^2) = 3 \tag{8}$$

(Example 1, Art. 77).

Solution. Two intermediate integrals of this equation were obtained in Art. 77,

$$y - 2x + p = \phi_1(y + x + 2q), \qquad y - 4x + 2p = \phi_2(y + 2x + 2q). \tag{9}$$

Because of the presence of q in each argument of the arbitrary functions, we cannot solve these relations for p and q and proceed as in Example 1. Instead we employ a parametric method.

Set the arguments of ϕ_1 and ϕ_2 in (9) equal, respectively, to parameters P and Q, so that the two intermediate integrals are given by the four relations

$$y + x + 2q = P, \tag{10}$$

$$y + 2x + 2q = Q, \tag{11}$$

$$y - 2x + p = \phi_1(P), \tag{12}$$

$$y - 4x + 2p = \phi_2(Q). \tag{13}$$

Solving (10) and (11) for x, and (12) and (13) for y, we find

$$x = Q - P, \tag{14}$$

$$y = 2\phi_1(P) - \phi_2(Q). \tag{15}$$

Also, from (12) and (13),

$$p = 2x - y + \phi_1(P) = 2x - \frac{y}{2} + \frac{1}{2}\phi_2(Q), \tag{16}$$

and from (10) and (11),

$$q = -\frac{x}{2} - \frac{y}{2} + \frac{P}{2} = -x - \frac{y}{2} + \frac{Q}{2}. \tag{17}$$

By trial, it will be found that for $dz = p\,dx + q\,dy$ to be integrable, we must combine either the first value of p and the second of q, or the second of p and the first of q. Arbitrarily choosing the former pair, and using, where necessary,

$$dx = dQ - dP, \qquad dy = 2\phi_1'(P)\,dP - \phi_2'(Q)\,dQ,$$

obtained from (14) and (15), we have

$$dz = 2x\,dx - y\,dx + \phi_1(P)\,dQ - \phi_1(P)\,dP - x\,dy - \frac{y}{2}\,dy + Q\phi_1'(P)\,dP - \frac{Q}{2}\phi_2'(Q)\,dQ.$$

Upon integration, this leads to

$$z = x^2 - xy - \frac{y^2}{4} + Q\phi_1(P) - \int \phi_1(P)\,dP - \frac{1}{2}\int Q\phi_2'(Q)\,dQ.$$

Integration of the last term by parts yields

$$\int Q\phi_2'(Q)\,dQ = Q\phi_2(Q) - \int \phi_2(Q)\,dQ.$$

Hence, introducing new arbitrary functions $f(P)$ and $g(Q)$, such that $f'(P) = \phi_1(P)$ and $g'(Q) = \phi_2(Q)$, we get

$$\begin{aligned} z &= x^2 - xy - \frac{y^2}{4} + Qf'(P) - f(P) - \frac{Q}{2}g'(Q) + \tfrac{1}{2}g(Q) \\ &= x^2 - xy - \frac{y^2}{4} + \frac{Q}{2}[2f'(P) - g'(Q)] - f(P) + \tfrac{1}{2}g(Q). \end{aligned}$$

Inserting these new symbols also in (14) and (15), and accordingly simplifying the expression in brackets above, we have finally

$$x = Q - P, \qquad y = 2f'(P) - g'(Q),$$
$$z = x^2 - xy - \frac{y^2}{4} + \frac{yQ}{2} - f(P) + \tfrac{1}{2}g(Q). \tag{18}$$

These three parametric equations, containing two arbitrary functions of the parameters P and Q, furnish us with a solution of (8).

Example 4. Solve the equation

$$x^2r + (q - xy)s - 2pt - x(rt - s^2) = yq - 2xp \tag{19}$$

(Example 2, Art. 77).

Solution. Monge's method led to only one intermediate integral of this equation,

$$xp + z = \phi(xy - q). \tag{20}$$

Here q appears in the argument of the arbitrary function; if we use the alternative inverse form, $xy - q = \psi(xp + z)$, p is present in the argument of the arbitrary function ψ. If p or q or both occur in this way, Charpit's method is not effective; consequently, when only one intermediate integral is on hand, it is, in general, a practical impossibility to obtain a solution involving two arbitrary functions.

We therefore particularize the arbitrary function so that the necessary integrations can be effected. If we take for ϕ a linear function of its argument, we get

$$xp + z = a(xy - q) + b,$$

or

$$xp + aq = axy + b - z, \tag{21}$$

where a and b are arbitrary constants. This is a Lagrange equation; it is left to the student to solve (21), thereby obtaining as a solution of (19),

$$z = \frac{axy}{2} - \frac{a^2x}{4} + b + e^{-y/a}f(xe^{-y/a}), \tag{22}$$

which involves one arbitrary function f in addition to the two arbitrary constants a and b.

EXERCISES

Complete the solution of the exercises of Arts. 76 and 77 by integrating the following intermediate integrals.

1. $p = \phi(z)$.
2. $q - p = \phi_1(z)$, $q = p\phi_2(x + y)$.
3. $p = q\phi(z)$.
4. $xp + xy^2 = \phi(xy)$.
5. $p = 2xy^2 + \phi(y - x)$.
6. $q = \log x + \phi(x^2 + y)$.
7. $p = e^y + x\phi(x^2 - y^2)$.
8. $yq = 3xy^3 + \phi(y/x)$.

9. $p - q = \phi_1(x + z)$, $p + 1 = q\phi_2(x + y)$.

10. $xp + z = x\phi(x/y)$.

11. $p + z = y^2e^y + e^y\phi(x - y^2)$.

12. $1 + p + q = x\phi_1(y + z)$, $x(1 + q) = (1 + p + q)\phi_2(x - y)$.

13. $q = xp\phi_1(xz)$, $xp + z = q\phi_2(z)$.

14. $(q - xp)e^y = q\phi(z)$.

15. $q - 1 = p\phi_1(x + z)$, $p + 1 = q\phi_2(y - z)$.

16. $p + 3y = \phi_1(2q + x)$, $2p + y = \phi_2(q + 3x)$.

17. $p + 2y = \phi_1(q - x)$, $p - y = \phi_2(q + 2x)$.

18. $p - 2x + y = \phi_1(q - x - y)$, $p - 2x - y = \phi_2(q + x - y)$.

19. $p + x + y = \phi(q + x + y)$.

20. $p - 2x + y = \phi(q + x - 2y)$.

21. $p + x = \phi_1(q - x - 2y)$, $p + x - y = \phi_2(q - 2y)$.

22. $p - 2x = \phi(q + y)$.

23. $p + y^2 = \phi(q - x^2)$.

24. $yp + x = \phi(q + y)$.

25. $xp - q = \phi(yq + p)$.

79. Poisson's method. In this and the following article we shall discuss other methods of solving second order partial differential equations. These methods will often be found effective for some types of equations to which Monge's methods do not apply.

Consider first an equation of the form

$$P = (rt - s^2)^k Q, \tag{1}$$

where P involves p, q, r, s, and t and is algebraically homogeneous in r, s, t; Q is any function of x, y, z, p, q, r, s, and t that is finite when $rt - s^2 = 0$; and k is a positive constant. Poisson's method, applicable to such an equation, is to seek an intermediate integral containing only p and q. If we set

$$q = \phi(p), \tag{2}$$

where ϕ is a function to be determined, we get by differentiation with respect to x and y in turn,

$$s = r\phi'(p), \qquad t = s\phi'(p) = r[\phi'(p)]^2. \tag{3}$$

Evidently, then, $rt - s^2 = 0$, and (1) reduces to

$$P = 0. \tag{4}$$

Moreover, because of the homogeneity of P with regard to r, s, and t, substitution for q, s, and t of the values given by (2) and (3) will yield the product of some power of r and a factor containing only p, ϕ, and ϕ'. Thus (4) reduces to two equations,

$$r = 0 \tag{5}$$

and

$$F[p, \phi(p), \phi'(p)] = 0. \tag{6}$$

From (5) we get, upon integration, $p = f(y)$ and

$$z = xf(y) + g(y). \tag{7}$$

But f and g cannot be arbitrary functions of y. For, since $q = xf'(y) + g'(y)$, (2) imposes the condition

$$xf'(y) + g'(y) = \phi[f(y)]. \tag{8}$$

Consequently $f'(y)$ must vanish, or $f = a_1$, a constant, and therefore also, by (8),

$$g'(y) = \phi(a_1) = b_1,$$

say, so that $g(y) = b_1y + c_1$. Thus (2) and (5) together yield the solution

$$z = a_1x + b_1y + c_1, \tag{9}$$

containing three arbitrary constants. Clearly (9) will be a solution of any equation of type (1).

On the other hand, if we solve the ordinary first order equation (6) for $\phi(p)$, obtaining a relation involving one arbitrary constant, substitution in (2) will give us a first order partial differential equation to which the methods of Chapters IV and V may be applied. We thereby may obtain a solution involving an arbitrary function in addition to the arbitrary constant.

Example. Solve the equation

$$16r^2 + s^2 + t^2 - 9rt = 0.$$

Solution. This equation may be written in the form (1); we have as an equivalent expression,

$$(4r - t)^2 = rt - s^2.$$

Substituting $q = \phi(p)$, this becomes

$$r^2(4 - \phi'^2)^2 = 0,$$

whence, dropping the factor r^2,

$$\phi'(p) = \pm 2,$$

$$q = \phi(p) = \pm 2p + a,$$

where a is a constant. By Lagrange's method we then have

$$\mp 2p + q = a,$$

$$\frac{dx}{\mp 2} = \frac{dy}{1} = \frac{dz}{a},$$

which easily yield the solutions

$$z = ay + f(x + 2y), \qquad z = ay + g(x - 2y),$$

where f and g are arbitrary functions of their respective arguments. It is apparent that each of these includes (9) as a special case.

EXERCISES

Solve each of the following equations by Poisson's method.

1. $4r - t = (rt - s^2)^2$.
2. $2r - 3s + t = 5(rt - s^2)$.
3. $r^2 + 2s^2 + 81t^2 = 20rt$.
4. $r - 2s + t = x(rt - s^2)$.
5. $p(r - 4s + 4t) = (rt - s^2)^3$.
6. $y(r^2 - t^2) + rt - s^2 = 0$.
7. $p^2r - 2pqs + q^2t = rt - s^2$.
8. $(4p^2 - 1)r + 4ps + t = 0$.
9. $q^2r - 3pqs + 2p^2t = pq(rt - s^2)$.
10. $5r^2t - rs^2 - 5s^2t + t^3 = 0$.

80. A general method. We consider now a general procedure, applicable to any second order partial differential equation

$$G(x, y, z, p, q, r, s, t) = 0 \tag{1}$$

in which G is a polynomial in r, s, and t.

We seek an intermediate integral,

$$u(x, y, z, p, q) = 0, \tag{2}$$

which may or may not contain arbitrary constants or an arbitrary function. Differentiating this with respect to x and y in turn, we have

$$u_x + u_z p + u_p r + u_q s = 0, \tag{3}$$

$$u_y + u_z q + u_p s + u_q t = 0. \tag{4}$$

If the given equation (1) possesses any intermediate integral (2), then (1) must be obtainable from the three relations (2)–(4). Consequently, when we substitute the expressions for r and t found from (3) and (4),

$$r = -\frac{u_x + u_z p + u_q s}{u_p}, \qquad t = -\frac{u_y + u_z q + u_p s}{u_q}, \tag{5}$$

in (1), the result must be an identity. Hence, either the total coefficient of each power of s (including the power zero) must vanish, or s and the group of terms free of s must both be zero.

With the former alternative, we get a set of equations involving x, y, z, p, q, and the first partial derivatives of the unknown function u with respect to these five quantities. This system of equations will then be of the form considered in Art. 52, and may be treated by the methods there discussed, or by simpler methods in some cases. In order that (2) be an intermediate integral of (1), at least one of the two first derivatives of z must be present in (2). In particular, therefore, if the system found as described above leads to the relations $u_p = 0$ and $u_q = 0$, indicating that u lacks both p and q, we conclude that the original second order equation does not possess an intermediate integral.

If the form of the given differential equation makes the supposition $s = 0$ tenable, whether or not there exists an intermediate integral, there may be a solution of the form $z = f(x) + g(y)$, and the character of the functions f and g should be determined.

When a second order equation possesses neither an intermediate integral nor a solution of the form $z = f(x) + g(y)$, we may try to find a solution of some other particular form, as in the following example.

Example 1. Apply the above method to the equation

$$rs + st = 1. \qquad (6)$$

Solution. Making the substitutions (5) and clearing of fractions, we get

$$u_p u_q = 0, \qquad u_x u_q + u_z u_q p + u_y u_p + u_z u_p q = 0, \qquad u_p^2 + u_q^2 = 0. \qquad (7)$$

From the first and third of these we see that $u_p = 0$ and $u_q = 0$. Hence (6) has no intermediate integral. Moreover, we evidently cannot have $s = 0$, so that our method fails to provide a solution.

If we tentatively assume a solution of the form

$$z = A + Bx + Cy + Dx^2 + Exy + Fy^2,$$

substitution in (6) yields the relation $2E(D + F) = 1$. Thus we get the solution

$$z = A + Bx + Cy + Dx^2 + \frac{xy}{2(D + F)} + Fy^2, \qquad (8)$$

containing five arbitrary constants.

Other solutions, involving higher powers of x and y, may be similarly obtained. In some cases it may be necessary or desirable to assume a solution in the form of an infinite power series, but we shall not attempt these processes here.

Example 2. Apply the above method to the equation

$$rs - xr - ys + p = 0. \qquad (9)$$

Solution. Replacing r by its expression (5) and clearing of fractions, we obtain

$$- u_x s - u_z ps - u_q s^2 + xu_x + xu_z p + xu_q s - yu_p s + pu_p = 0, \quad (10)$$

which leads to the system

$$xu_x + xu_z p + pu_p = 0, \qquad u_x + u_z p - xu_q + yu_p = 0, \qquad u_q = 0. \quad (11)$$

We first treat this system by the method of Art. 52, to illustrate the general procedure; we shall then solve our problem by a simpler process.

Employing the usual notation,

$$x = x_1,\ y = x_2,\ z = x_3,\ p = x_4,\ q = x_5; \quad p_j = \frac{\partial u}{\partial x_j} \quad (j = 1, 2, \cdots, 5),$$

the system may be written as

$$\begin{aligned} F &\equiv x_1p_1 + x_1x_4p_3 + x_4p_4 = 0, \\ F_1 &\equiv p_1 + x_4p_3 + x_2p_4 - x_1p_5 = 0, \\ F_2 &\equiv p_5 = 0. \end{aligned} \quad (12)$$

Then (Art. 52)

$$(F, F_1) = p_1 + x_4p_3 + x_1p_5 + (x_1p_3 + p_4)x_2 - x_4p_3. \quad (13)$$

Reducing this by means of the relations $F_1 = 0$ and $F_2 = 0$, we have

$$(F, F_1) = x_1x_2p_3 - x_4p_3 = (x_1x_2 - x_4)p_3. \quad (14)$$

In order that the system shall have a common solution, we must have $(F, F_1) = 0$. If $p_3 = 0$, $F = 0$ and $F_1 = 0$ reduce to

$$F = x_1p_1 + x_4p_4 = 0, \qquad F_1 = p_1 + x_2p_4 = 0. \quad (15)$$

Now since $p_5 \equiv u_q = 0$ by $F_2 = 0$, $p_4 \equiv u_p$ must be different from zero if the original equation is to have an intermediate integral. Consequently the relations (15) can hold only if the determinant of that system, $x_1x_2 - x_4$, is zero. Hence $(F, F_1) = 0$ only if $x_1x_2 - x_4 = 0$, that is, only if

$$p = xy. \quad (16)$$

This is an intermediate integral; integration gives us

$$z = \frac{x^2y}{2} + f(y), \quad (17)$$

where f is arbitrary. It is easily verified that this is a solution of the given differential equation.

In this case it is easier to solve the problem as follows. Using the third of equations (11), the other two become

$$xu_x + xu_z p + pu_p = 0, \qquad u_x + u_z p + yu_p = 0.$$

From these we get

$$u_x + u_z p = -\frac{pu_p}{x} = -yu_p.$$

Again, since $u_q = 0$, $u_p \neq 0$. Therefore the last equality yields

$$p = xy,$$

as before, whence the solution (17) is found.

Now the possibility that $s = 0$ must also be examined. Setting

$$z = f(x) + g(y), \tag{18}$$

obtained from $s = 0$, we have

$$p = f', \quad q = g', \quad r = f'', \quad s = 0, \quad t = g'',$$

and substitution in (9) gives us

$$-xf'' + f' = 0,$$

or

$$x\frac{df'}{dx} - f' = 0.$$

This readily yields $f(x) = ax^2 + b$, where a and b are arbitrary constants. Substituting in (18), and absorbing the constant b in the arbitrary function $g(y)$, we then get the additional solution

$$z = ax^2 + g(y). \tag{19}$$

EXERCISES

Solve each of the following equations.

1. $rs + t = 1$.
2. $st - r = 0$.
3. $st - s = p$.
4. $s^2 - qr = 0$.
5. $r^2 - ps = 0$.
6. $s^2 - t = 2$.
7. $rs - s^2 + s = 1$.
8. $rt + s^2 = 0$.
9. $s^2 - r - t = 0$.
10. $rs + 2s^2 + st = r - t$.

81. Geometric problems. The general procedure for solving a geometric problem, in which a non-linear partial differential equation is concerned, is similar to that discussed in Art. 70. The geometric properties involved must, of course, suit the type of solution obtainable in each given case.

When dealing with developable surfaces (Art. 26), Poisson's method, when effective, is particularly useful. For, it was found in Art. 29 that p and q are functionally related for these surfaces, and Poisson's method takes just such a relation, $q = \phi(p)$, as its starting point.

Example. Find the equations of developable surfaces satisfying the differential equation

$$rt - s^2 - r + t = 0, \qquad (1)$$

and passing through the circle $x^2 + z^2 = 1$, $y = 0$ and through the point (0, 1, 2).

Solution. Setting $q = \phi(p)$, so that $s = r\phi'$, $t = s\phi' = r\phi'^2$, we get

$$r(1 - \phi'^2) = 0. \qquad (2)$$

Evidently we may discard the factor r, since $r = 0$ leads only to planes (Art. 79), which cannot satisfy the given geometric conditions. Hence we have

$$\phi'(p) = 1, \qquad q = \phi(p) = p + a; \qquad (3)$$

or

$$\phi'(p) = -1, \qquad q = \phi(p) = -p + a. \qquad (4)$$

From the respective Lagrange equations (3) and (4) we easily find two families of developable surfaces satisfying (1):

$$z = ay + f(x + y), \qquad (5)$$

$$z = ay + g(x - y), \qquad (6)$$

where f and g are arbitrary functions and a is an arbitrary constant.

We now apply the first geometric condition, $z = \pm\sqrt{1 - x^2}$, $y = 0$, to equations (5) and (6); this gives us

$$\pm\sqrt{1 - x^2} = f(x), \qquad \pm\sqrt{1 - x^2} = g(x),$$

whence $f(x + y) = \pm\sqrt{1 - (x + y)^2}$ and $g(x - y) = \pm\sqrt{1 - (x - y)^2}$, and

$$z = ay \pm \sqrt{1 - (x + y)^2}, \qquad z = ay \pm \sqrt{1 - (x - y)^2}. \qquad (7)$$

Finally, if each surface is to pass through the point (0, 1, 2), we get in each case $a = 2$, and therefore

$$(x + y)^2 + (z - 2y)^2 = 1, \qquad (8)$$

$$(x - y)^2 + (z - 2y)^2 = 1 \qquad (9)$$

are two developable surfaces meeting all the requirements. Equations (8) and (9) represent elliptic cylinders.

EXERCISES

1. Show that the differential equation $xq^2r - 2xpqs + xp^2t = xp^2q - 2pq^2$ is satisfied by all cylinders with elements parallel to the x-axis.

2. Show that the traces in the xz-plane of surfaces satisfying the differential equation $xr - ys + p = y^2$ are straight lines parallel to the x-axis.

3. Show that surfaces satisfying the differential equation $(q+1)^2r - 2p(q+1)s + p^2t = 0$ are generated by straight lines parallel to the plane $y + z = 0$.

4. Show that the surfaces whose sections by planes parallel to the xz-plane are circles through the y-axis, satisfy the equation $(x^2 + z^2)r = 2(xp - z)(1 + p^2)$.

5. Show that the differential equation $xq^2r - q(z + 2xp)s + p(xp + z)t + pq^2 = 0$ is satisfied by surfaces whose traces in the xy-plane are straight lines parallel to the x-axis.

6. Find the equation of the surface satisfying the differential equation $q(1+q)r - p(1+2q)s + p^2t = 0$ and passing through the line $y + z = 1$, $x = 0$ and the parabola $y^2 = x + 1$, $z = 0$.

7. Find the equation of the surface satisfying the differential equation $x(1+q)r + x(1 - p + q)s - xpt = (1 + p + q)(1 + q)$ and passing through the line $y = x$, $z = 0$ and the x-axis.

8. Find the equations of developable surfaces satisfying the differential equation $4r - t = (rt - s^2)^2$ and passing through the parabola $x = z^2$, $y = 0$ and the point $(1, -4, 5)$.

9. Find the equation of the surface satisfying the differential equation $2rt + s^2 = 0$ and passing through the parabola $z = y^2$, $x = 0$ and the point $(1, -1, 3)$.

10. Find the equation of the surface satisfying the differential equation $2rs + 5s^2 + 2st = 4r - t$ and passing through the circle $y^2 + z^2 = 1$, $x = 0$ and the point $(2, 1, 1)$.

ANSWERS TO EXERCISES

Arts. 2–3. Page 3

1. $y^2 + 6y - 2x^2 = c.$
2. $9x^2 + 6y + 2 \log (1 - 3y) = c.$
3. $(2 - x)(\sec y + \tan y) = c.$
4. $y\sqrt{1 - x^2} - x\sqrt{1 - y^2} = c.$
5. $(x - 1)e^x + e^{-y} = c.$
6. $\log^3 y = cx.$
7. $3\sqrt{1 + y} + (2 - x)^{3/2} = c.$
8. $2 \cos x + \log (\csc y + \cot y) = c.$
9. $2xy + 3x^2 = c.$
10. $xy - y^3 = c.$
11. $3x^3 + y = cx.$
12. $x^2y + cxy + 1 = 0.$
13. $y = x \tan (x + c).$
14. $\log \dfrac{x + y}{x - y} = 2x + c.$
15. $y^2 - 2ye^x = c.$
16. $y \sin x + x^2 = c.$
17. $y^2 - x^3 = cx.$
18. $3x^2 + \tan y = cx.$
19. $e^x + 4y^2 = cy.$
20. $y^2 - \log x = cy.$

Arts. 4–5. Pages 6–7

1. $y = ce^{2x} - e^x.$
2. $y = 2x - 2 + ce^{-x}.$
3. $y = ce^{3x} - \dfrac{1}{6}(\sin 3x + \cos 3x).$
4. $3x^4y - x^3 = c.$
5. $2x^3y - \log^2 x = c.$
6. $y = x(\log \log x + c).$
7. $x = y - 2 + ce^{-y/2}.$
8. $x \cos y - y = c.$
9. $y^2 = e^{-x} + ce^x.$
10. $y(4x - 1 + ce^{-4x}) = 16.$
11. $4y \log x + x = cy.$
12. $y^3 + 9x^3 \log x = cx^3.$
13. $\sin \dfrac{y}{x} - \log x = c.$
14. $y + \sqrt{x^2 + y^2} = cx^2.$
15. $y \log \dfrac{y}{x} - y = x \log x + cx.$
16. $\tan \dfrac{y}{2x} - \log x = c.$
17. $y = x \log (x + y) + cx.$
18. $x = y \sin (\log y + c).$
19. $x^2 - 2xy - y^2 - 12x + 4y = c.$
20. $x^2 - 4xy + 4y^2 + 6x + 10y = c.$
21. $2xy = 3(x^2 - 4).$
22. $5y = 2x^3 - 60\sqrt{x}.$
23. $y \cos x = x + \sin x \cos x - 1 - \pi.$
24. $2y = 13e^{5x} - e^{3x}.$
25. $(x + y)^3(x - y) = 1.$
26. $(x + 3y)^3(x - 2y)^2 = 64x^5y^5.$
27. $\log 4 - 1.$
28. $\frac{3}{2}.$
29. $-\frac{1}{243}.$
30. $-1 - \log 3.$

Arts. 6–8. Page 13

1. $x^2y - y^2 + c(y - ye^x + x^2e^x) + c^2e^x = 0.$
2. $y^2 = x^2 \sin^2 (\log x + c); y = \pm x.$
3. $4x^5 - 25x^2y^2 + 10cxy - c^2 = 0.$
4. $y^2 - 4x + 2cxy + c^2x^2 = 0.$
5. $x = 2p + 2 \log (p - 1) + c, \quad y = p^2 + 2p + 2 \log (p - 1) + c; y = x + 1.$
6. $y^2 = 2cx - c^2; \quad y = \pm x.$
7. $x = c(p + 1)e^p, \quad y = cp^2e^p.$

8. $x = -2p - 6\log(p-2) + c, \quad y = 2c - 12\log(p-2) - p^2 - 6p - 1;$
$y = 2x - 9.$

9. $y^2 = 2cx + c^2 + 1; \quad x^2 + y^2 = 1.$

10. $4 + cy + c^2x = 0; \quad y^2 = 16x.$

11. $x = 2p - 3p^2, \quad y = p^2 - 2p^3 + c.$

12. $1 + cy - c^2e^x = 0.$

13. $x = cpe^{-p^2/2}, \quad y = c(p^2+1)e^{-p^2/2}.$

14. $y = cx + c^2; \quad x^2 + 4y = 0.$

15. $y^2 + 2c(4 - xy) + c^2x^2 = 0; \; xy = 2.$

16. $y = cx - c^3; \quad 4x^3 = 27y^2.$

17. $1 + c(y-1) - c^2x = 0; \quad (y-1)^2 + 4x = 0.$

18. $1 + 3y + c(3x + 2y) + 2c^2x = 0; \quad (3x - 2y)^2 = 8x.$

19. $x = c(p+1)e^{p/2}/\sqrt{p}, \quad y = c(p-1)\sqrt{p}\,e^{p/2}; \quad y = 0.$

20. $x = 2p - 2 + ce^{-p}, \quad y = p^2 - 2 + c(p+1)e^{-p}.$

Art. 9. Pages 16–17

1. $y = (x+1)e^x + 2.$

2. $2y = x^4 + 4\log x + 4.$

3. $x^5 - 4xy + 31x + 4 = 0.$

4. $y = x - 4\arctan\dfrac{x}{2} - 2.$

5. $4y = 2x^2 + 2x + \log(2x-1) - 36.$

6. $3y = x^6 - 3x + 2.$

7. $y = 3e^{2x} - 5e^{-2x}.$

8. $y = 2\sin 3x - \cos 3x.$

9. $y(x-1)^2 = 1.$

10. $y\sqrt{y} = 3\sqrt{2}(1-x).$

11. $y^2(2x+1) + 3 = 0.$

12. $y = 2(x+5)^2.$

13. $\log(y+4) = 2x + 2.$

14. $y^3(3x - 8) = 4.$

15. $y = c_1e^{kx} + c_2e^{-kx}.$

16. $y = c_1\sin kx + c_2\cos kx.$

17. $(x + c_1)^2 + (y + c_2)^2 = 4.$

18. $(x + c_1)^2 + y^2 = c_2.$

19. $x^2 - (y + c_1)^2 = c_2.$

20. $y^2 = 2(x + c_1)^3 + c_2.$

Art. 11. Page 19

1. $y = c_1e^{3x} + c_2e^{-3x}$

2. $y = c_1\sin 3x + c_2\cos 3x.$

3. $y = c_1 + c_2e^{-5x}.$

4. $y = c_1e^{7x} + c_2e^{-4x}.$

5. $y = (c_1 + c_2x)e^{-3x}.$

6. $y = c_1 + c_2x + c_3e^{-6x}.$

7. $y = (c_1 + c_2x + c_3x^2)e^{2x}.$

8. $y = c_1e^{2x} + e^{-x}(c_2\sin\sqrt{3}x + c_3\cos\sqrt{3}x).$

9. $y = c_1e^{2x} + c_2e^{-2x} + c_3\sin 2x + c_4\cos 2x.$

10. $y = (c_1 + c_2x)e^{2x} + (c_3 + c_4x)e^{-2x}.$

11. $y = c_1 + (c_2 + c_3x + c_4x^2)e^x.$

12. $y = (c_1 + c_2x + c_3x^2 + c_4x^3)e^{x/2}.$

13. $y = e^x(c_1\sin x + c_2\cos x) + e^{-x}(c_3\sin x + c_4\cos x).$

14. $y = c_1 + c_2e^{3x} + c_3e^{-3x} + c_4\sin 2x + c_5\cos 2x.$

15. $y = 2e^{2x} - 3e^{-2x}.$

16. $y = -3\cos 4x.$

17. $y = 5.$

18. $y = 2xe^{-x}.$

19. $y = 1 - 3e^x + e^{-4x}.$

20. $y = e^x - 3\sin x.$

Art. 12. Page 24

1. $y = c_1e^{2x} + c_2e^{-2x} + 4x - 3x^2.$

2. $y = c_1e^x + c_2e^{-2x} + 6 - e^{-x}.$

3. $y = c_1 + c_2e^{-x} + (2x+1)e^x/2.$

4. $y = c_1\sin 3x + c_2\cos 3x + \sin 2x.$

5. $y = c_1 + c_2e^{3x} + 3x - 7e^{2x}.$

6. $y = (c_1 - \frac{1}{3}x + x^2)e^{2x} + c_2e^{-4x}.$

7. $y = c_1\sin x + (c_2 - 3x)\cos x.$

8. $y = c_1 + c_2e^{3x} + c_3e^{-3x} + 5x^2 - e^x.$

9. $y = c_1 + c_2x + c_3e^{-x} + 8x^2 - xe^{-x}.$

10. $y = c_1 + (c_2 + 2x)\sin 2x + c_3\cos 2x - 5x.$

11. $y = (c_1 - 3x)e^x + c_2e^{-x} + c_3 \sin x + (c_4 + x) \cos x.$
12. $y = c_1 \sin x + c_2 \cos x - 2 \cos x \log (\sec x + \tan x).$
13. $y = c_1 \sin 2x + c_2 \cos 2x - 2 \sin 2x \log (\csc 2x + \cot 2x).$
14. $y = e^x(c_1 + c_2x - 3 \log x).$ **15.** $y = e^{-2x}(c_1 \sin x + c_2 \cos x - \cos 2x).$
16. $y = c_1 \sin 2x + c_2 \cos 2x + 4x \sin 2x + 2 \cos 2x \log \cos 2x.$
17. $y = e^{-x}[c_1 \sin 3x + c_2 \cos 3x - \cos 3x \log (\sec 3x + \tan 3x)].$
18. $y = e^{3x}(c_1 \sin x + c_2 \cos x + 5 \sin x \log \sin x - 5x \cos x).$
19. $y = \left(c_1 + \sqrt{2} \log \dfrac{1 + \sqrt{2} \sin x}{1 - \sqrt{2} \sin x}\right) \sin x + \left(c_2 - \sqrt{2} \log \dfrac{1 + \sqrt{2} \cos x}{1 - \sqrt{2} \cos x}\right) \cos x.$
20. $y = c_1 + (c_2 + x)e^{2x} + c_3e^{-2x} + e^x + e^{-x} - 2x - (e^x - e^{-x})^2 \log (e^x + 1).$

Arts. 13–14. Page 27

1. $y = c_1 + c_2 \log x + \log^3 x - 1/x.$ **2.** $y = c_1 + c_2x^4 - x^2 + 2x.$
3. $y = c_1 \sin (\log x) + c_2 \cos (\log x) - \sin (2 \log x).$
4. $y = c_1x + c_2x^2 - 2x \log x - 4.$
5. $y = c_1 \sin (2 \log x) + c_2 \cos (2 \log x) + (\log x) \sin (2 \log x).$
6. $y = c_1/x^2 + c_2/x^3 + 4(\log^2 x)/x^2.$
7. $y = c_1 + c_2 \log x + c_3x^4 + 3 \log^3 x.$
8. $y = c_1x^2 + [c_2 \sin (\sqrt{3} \log x) + c_3 \cos (\sqrt{3} \log x)]/x + 5x^2 \log x - 3.$
9. $y = [c_1 + c_2 \log (2x + 1)] (2x + 1) + 6 - 3 \log^2 (2x + 1).$
10. $y = c_1 \sin[\frac{1}{3} \log (3x - 2)] + c_2 \cos[\frac{1}{3} \log (3x - 2)] + 6x - 9.$
11. $y = -\frac{3}{2}c_1e^x + c_2e^{-4x} - 2e^{-x}, \quad z = c_1e^x + c_2e^{-4x} + 5.$
12. $y = c_1e^{-2x} + c_2e^{22x} + 4x - 2, \quad z = \frac{7}{2}c_1e^{-2x} - \frac{5}{2}c_2e^{22x} + 6x + 1.$
13. $y = c_1e^{x/2} + c_2e^{-x} - 2e^{2x}, \quad z = -5c_1e^{x/2} - 2c_2e^{-x} + 4e^{2x}.$
14. $y = 3 \sin x, \quad z = -\cos x.$
15. $y = c_1e^x + c_2 \sin x + c_3 \cos x + 3x^2, \quad z = -2c_1e^x - (3c_2 - c_3) \sin x$
$- (c_2 + 3c_3) \cos x + 4x - 1.$
16. $y = x^2 + c_1, \quad z = -3x, \quad w = 6.$
17. $y = c_1e^{2x} + c_2e^{-x} \sin (\sqrt{3}x + c_3) - 1, \quad z = c_1e^{2x} + c_2e^{-x} \sin (\sqrt{3}x + c_3 + \frac{2}{3}\pi)$
$- 1, \quad w = c_1e^{2x} + c_2e^{-x} \sin (\sqrt{3}x + c_3 + \frac{4}{3}\pi) - 1.$
18. $y = c_1e^{-x} + c_2e^{-2x} + e^x, \quad z = \frac{1}{2}c_1e^{-x} + c_2e^{-2x} + 3, \quad w = c_2e^{-2x} - e^x.$
19. $y = -2c_1x + 4c_2/x^5 + 7 - 2x^2, \quad z = c_1x + c_2/x^5 + 3 + 8x^2.$
20. $y = c_1x^2 + c_2x^{2/7} + 3x^2 \log x, \quad z = \frac{1}{2}c_1x^2 + \frac{13}{14}c_2x^{2/7} - x^2 \log x.$

Art. 15. Pages 29–30

2. $y = c_1x + c_2e^x.$ **3.** $y = c_1x \sin x + c_2x \cos x.$ **4.** $y = (c_1 + c_2 \sqrt{x})e^x.$

5. $$y = c_1\left(1 - \frac{x}{1!} + \frac{x^2}{4 \cdot 2!} - \frac{x^3}{7 \cdot 4 \cdot 3!} + \frac{x^4}{10 \cdot 7 \cdot 4 \cdot 4!} - \cdots\right) + c_2x^{2/3}\left(1 - \frac{x}{5 \cdot 1!} + \frac{x^2}{8 \cdot 5 \cdot 2!} - \frac{x^3}{11 \cdot 8 \cdot 5 \cdot 3!} + \cdots\right), |x| < \infty.$$

6. $y = (c_1 + c_2 \sqrt{x}) \sqrt{1 - x^2}.$

7. $y = c_1 x^n \left[1 - \frac{x^2}{4(1+n)} + \frac{x^4}{4\cdot 8(1+n)(2+n)} - \frac{x^6}{4\cdot 8\cdot 12(1+n)(2+n)(3+n)} + \cdots\right]$
$+ c_2 x^{-n}\left[1 - \frac{x^2}{4(1-n)} + \frac{x^4}{4\cdot 8(1-n)(2-n)} - \frac{x^6}{4\cdot 8\cdot 12(1-n)(2-n)(3-n)} + \cdots\right]$,
$x \neq 0$.

8. $y = c_1\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \cdots\right]$
$+ c_2\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \cdots\right], |x| < 1$.

10. $y = c_1 \sin\frac{1}{x} + c_2 \cos\frac{1}{x}$.

Arts. 16–17. Pages 34–35

1. $r = 18xy^2, \quad s = 18x^2y + 4, \quad t = 6x^3 - 12y$.
2. $r = 8e^{3y} - 5y^2e^{-x}, \quad s = 24xe^{3y} + 10ye^{-x}, \quad t = 36x^2e^{3y} - 10e^{-x}$.
3. $r = -y^2/(x^2 - y^2)^{3/2} - y^2 \sin xy, \quad s = xy/(x^2 - y^2)^{3/2} - xy \sin xy + \cos xy$,
$t = -x^2/(x^2 - y^2)^{3/2} - x^2 \sin xy$.
4. $r = t = \frac{1}{2}\sin(x+y) - \frac{25}{2}\sin 5(x-y), \quad s = \frac{1}{2}\sin(x+y) + \frac{25}{2}\sin 5(x-y)$.
5. $r = 2y^2(x^2y^2 - xy - 1)/(1 + x^2y^2)^2, \quad s = (1 - 4xy - x^2y^2)/(1 + x^2y^2)^2$,
$t = 2x^2(x^2y^2 - xy - 1)/(1 + x^2y^2)^2$.
6. $r = 6y \sec^2 x \tan x, \quad s = 4 \sec y \tan y + 3 \sec^2 x, \quad t = 4x \sec y (\sec^2 y + \tan^2 y)$.
7. $w_{xx} = 10y^2z^2, \quad w_{yy} = 10x^2z^2, \quad w_{zz} = 10x^2y^2, \quad w_{xy} = 20xyz^2 - 3z$,
$w_{xz} = 20xy^2z - 3y, \quad w_{yz} = 20x^2yz - 3x$.
8. $w_{xx} = -yz \sin x, \quad w_{yy} = -zx \sin y, \quad w_{zz} = -xy \sin z$,
$w_{xy} = \sin z + z \cos x + z \cos y, \quad w_{xz} = y \cos z + y \cos x + \sin y$,
$w_{yz} = x \cos z + \sin x + x \cos y$.
9. $w_{xx} = -2y^2ze^{xy}, \quad w_{yy} = -2x^2ze^{xy}, \quad w_{zz} = 2xye^z$,
$w_{xy} = 2e^z - 2xyze^{xy} - 2ze^{xy}, \quad w_{xz} = 2ye^z - 2ye^{xy}, \quad w_{yz} = 2xe^z - 2xe^{xy}$.
10. $w_{xx} = 2z/x^3, \; w_{yy} = 2x/y^3, \; w_{zz} = 2y/z^3, \; w_{xy} = -1/y^2, \; w_{xz} = -1/x^2$,
$w_{yz} = -1/z^2$.
11. $p = -1, q = -1$.
12. $p = -(ye^z + ze^y + yze^x)/(xye^z + ye^x + xe^y)$,
$q = -(xe^z + ze^x + xze^y)/(xye^z + ye^x + xe^y)$.
13. $p = -\cot x \tan z, \quad q = -\cot y \tan z$.
14. $p = -(2z\sqrt{xy} + 2z\sqrt{xz} + z\sqrt{yz})/(x\sqrt{xy} + 2x\sqrt{yz} + 2x\sqrt{xz})$,
$q = -(2z\sqrt{xy} + 2z\sqrt{yz} + z\sqrt{xz})/(y\sqrt{xy} + 2y\sqrt{xz} + 2y\sqrt{yz})$.
15. $p = -(z \log z)/(x \log x), \quad q = -(z \log z)/(y \log y)$.
23. 35°. **24.** 75° 58′. **25.** $2y + 3z = 8, \quad x = 3$.

Arts. 18–20. Pages 41–42

1. $2u(8x - 3y) + (3u^2 - 1)(10y - 3x)$.
2. $(\cos y - y \sin x)e^u - (\cos x - x \sin y)e^{-u}$.
3. 6. **4.** $y(e^z - ze^{xy}) - x(e^z - ze^{xy})/u^2 + (xye^z - e^{xy})/u$.
5. $p = x^3 + 6x^2y + 6xy^2 + 4y^3, \quad q = -5x^3 + 3x^2y - 30xy^2 + 2y^3$.
6. $p = (2ye^{2x} - 3e^{-y})/(u + v) + (e^{2x} + 3xe^{-y})/(u - v)$,
$q = (2ye^{2x} - 3e^{-y})/(u + v) - (e^{2x} + 3xe^{-y})/(u - v)$.

7. $p = 8u(y \cos xy - \sin y) + 3(x \cos xy - x \cos y)$,
$q = 2(y \cos xy - \sin y) - 2v(x \cos xy - x \cos y)$.
8. $w_u = (2xyz - 1)(yze^{u+v} + xze^{u-v} + xyve^{uv})$,
$w_v = (2xyz - 1)(yze^{u+v} - xze^{u-v} + xyue^{uv})$.
9. $(x^2 - ay)/(ax - y^2)$. 10. $-(\sin y + y \cos x)/(x \cos y + \sin x)$.
11. $p = -[\sin z + z \cos (x - y)]/[(x + y) \cos z + \sin (x - y)]$,
$q = -[\sin z - z \cos (x - y)]/[(x + y) \cos z + \sin (x - y)]$.
12. $p = -(2xe^z + yze^{xy})/[(x^2 + y^2)e^z + e^{xy}]$,
$q = -(2ye^z + xze^{xy})/[(x^2 + y^2)e^z + e^{xy}]$.
13. $(f_xg_y - f_yg_x)/g_y$. 18. $u_x = g_v/(f_ug_v - f_vg_u)$, $v_y = f_u/(f_ug_v - f_vg_u)$.
20. $u_y = -(F_yG_v - F_vG_y)/(F_uG_v - F_vG_u)$,
$v_x = -(F_uG_x - F_xG_u)/(F_uG_v - F_vG_u)$.

Arts. 21–23. Page 49

1. $2x - y + 4z = 21$; $\dfrac{x-2}{2} = \dfrac{y+1}{-1} = \dfrac{z-4}{4}$.
2. $9x - 4y + 35 = 0$; $4x + 9y = 6$, $z = 5$.
3. $2x - 3y - z = 3$; $\dfrac{x-4}{2} = \dfrac{y-2}{-3} = \dfrac{z+1}{-1}$.
4. $x = 2$; $y = 0$, $z = 0$.
5. $x - 2y - z = 0$; $\dfrac{x+1}{1} = \dfrac{y+1}{-2} = \dfrac{z-1}{-1}$.
6. $3x - 2y + 6z + 18 = 0$; $\dfrac{x+2}{3} = \dfrac{y-3}{-2} = \dfrac{z+1}{6}$. 7. $90°$.
10. $a = 10$, $b = -\frac{2}{3}$. 11. $4 - x = y - 2 = 2z + 2$. 12. $35° 16'$.
17. $\alpha = 90°$, $\beta = 135°$, $\gamma = 45°$. 18. $\alpha = 42° 2'$, $\beta = 68° 12'$, $\gamma = 56° 9'$.
19. $62° 59'$. 20. (a) $f'F' + g'G' + h'H' = 0$; (b) $f'/F' = g'/G' = h'/H'$.

Arts. 24–26. Pages 57–59

11. $x + y = 0$. 12. $x^2 + y^2 = 1$. 13. $x^2 + y^2 = 4$. 14. $x = 1$.
15. $x = 0$, $y = 0$. 16. $x^2 + y^2 = 1$, $x^2 + y^2 = 2$. 17. $4\pi^2x^2y^2 = 1$.
18. $x^{2/3} + y^{2/3} = 1$. 19. $4x^2y^2 = 1$. 20. $x^2 - 2xy + y^2 - 2x - 2y + 1 = 0$.
21. $x^2 + y^2 = 1$. 22. $x^2 + 4yz = 0$. 23. $z^2 = 4xy$. 24. $(x + y)^2 + 4z = 0$.
25. $(1 + 2x + 2y - z)^2 = 8(x + y)$. 26. $xy + 4z = 0$. 27. $x^2 - y^2 = 2z$.
28. $z = \pm 1$. 29. $x^{1/2} + y^{1/2} + z^{1/2} = 1$. 30. $48\pi^2x^2y^2z^2 = 1$.

Art. 27. Pages 62–63

1. $z = xp + yq$. 2. $z = (x + y)p$, $z = (x + y)q$. 3. $xp + yq = 2z$.
4. $2xz = (x^2 + y^2)p$, $2yz = (x^2 + y^2)q$. 5. $4z = p^2 + q^2$.
6. $(x - y)(p^2 + q^2) = 2z(p - q)$. 7. $z^2(p^2 + q^2 + 1) = 1$.
8. $z^2p^2 + (y - x - zp)^2 + z^2 = 1$, $(x - y - zq)^2 + z^2q^2 + z^2 = 1$.
9. $xp = yq$. 10. $z = xp + yq$. 11. $z^2 = 1 + xzp + yzq$.

12. $xp + yq = 0$. **13.** $4pq = 1$. **14.** $p = q \tan (yp/z)$.
15. $r = 0, \quad s = 0, \quad t = 0$. **16.** $p = xr + ys, \quad q = xs + yt$.
17. $(p^2+q^2+1)(1+p^2)^2 = r^2, \quad p^2q^2(p^2 + q^2+1) = s^2, \quad (p^2+q^2+1)(1+q^2)^2 = t^2$.
18. $(1 + p^2)s = pqr, \quad (1 + q^2)s = pqt$. **19.** $s = 0$. **20.** $z = x_1p_1 + x_2p_2 + x_3p_3$.

Art. 28. Page 65

1. $z = xp$. **2.** $z = yq$. **3.** $(x + y)(xp - yq) = (x - y)z$.
4. $xy(p - q) = (x - y)z$. **5.** $xp + yq + z = 0$. **6.** $yp - xq = 0$.
7. $x(z - y)p + y(x - z)q = z(y - x)$. **8.** $x^2p - y^2q = (x - y)z$.
9. $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)$.
10. $(x + y)(x + 2z)p - (x + y)(y + 2z)q = (x - y)z$. **11.** $s = 1$.
12. $6r - 13s + 6t = 0$. **13.** $t = 0$. **14.** $x^2r - 2xp + 2z = 0$.
15. $zs = pq$. **16.** $t = xq$. **17.** $(1 + p)s = (1 + q)r$.
18. $y\dfrac{\partial^3 z}{\partial x^2 \partial y} = \dfrac{\partial^2 z}{\partial x^2}, x\dfrac{\partial^3 z}{\partial x \partial y^2} = \dfrac{\partial^2 z}{\partial y^2}$. **19.** $\dfrac{\partial^3 z}{\partial x^3} = 0$. **20.** $\dfrac{\partial^3 z}{\partial x^2 \partial y} = \dfrac{\partial^3 z}{\partial y^3}$.

Art. 29. Pages 68–69

1. $xp + yq = z$. **2.** $p = 0$. **3.** $xp + yq = z - 1$. **4.** $q = 0$.
5. $(y - z)p + (z - x)q = x - y$. **6.** $(x - y)^2(p^2 + q^2 + 1) = (p - q)^2$.
7. $(x^2 - 2xy + y^2 - z^2)(p^2 + q^2) + 2z^2pq + 2z(y - x)(p - q) = 0$.
8. $2z(xp + yq) + x^2 + y^2 - z^2 = 0$. **9.** $yp = xq$. **10.** $p^2 + q^2 = 1$.
11. $3p^2 - q^2 = 1$. **12.** $(xp + yq - z)(pq - p - q) + pq = 0$.
13. $(xp + yq - z)^3 + pq = 0$. **14.** $(xp + yq - z)^2(p^2q^2 + p^2 + q^2) = p^2q^2$.
15. $r = 0, \quad s = 0, \quad t = 0$. **16.** $pqr = (1 + p^2)s, \quad pqt = (1 + q^2)s$.
17. $s = 0$. **18.** $x^2r + 2xys + y^2t - 2xp - 2yq + 2z = 0$.
19. $pq(r - t) = (p^2 - q^2)s$.
20. $x^2pqr + (z^2 - x^2p^2 - y^2q^2)s + y^2pqt - 2pq(xp + yq - z) = 0$.

Arts. 35–37. Page 90

1. $\phi(2y + 3x, z - 2x) = 0$. **2.** $\phi(Ay - Bx, Az - Cx) = 0$.
3. $\phi(xy, yz) = 0$. **4.** $\phi(x^2 + y^2, xz) = 0$. **5.** $\phi(x^2 - y^2, z) = 0$.
6. $\phi\left(\dfrac{x - y}{xy}, \dfrac{x - z}{xz}\right) = 0$. **7.** $\phi\left(\dfrac{x + A}{y + B}, \dfrac{z + C}{y + B}\right) = 0$.
8. $\phi(2z - x^2, 2z - y^2) = 0$. **9.** $\phi(x^2 - y^2, y^2 - z^2) = 0$.
10. $\phi[(x - z)(z - y), z] = 0$. **11.** $\phi(x + y + z, x^3 + y^3 + z^3) = 0$.
12. $\phi\left(xy, \dfrac{z}{x + y}\right) = 0$. **13.** $\phi[x + y, z - x \log (x + y)] = 0$.
14. $\phi(x^2 + y^2, z^2 - x^4 - x^2y^2) = 0$. **15.** $\phi(y/x^2, 2 \log z - 2x - y) = 0$.
16. $\phi\left(\dfrac{x + y}{xy}, \dfrac{z}{xy}\right) = 0$. **17.** $\phi(x - 2y - z, x^2 + y^2 + z^2) = 0$.
18. $\phi(x + y + z, xyz) = 0$. **19.** $\phi(x + y, xyz) = 0$.
20. $\phi(x^2 + y^2 + z^2, xyz) = 0$.

Arts. 38–39. Pages 94–95

1. Straight lines. **2.** Circles. **3.** $(x^2+y^2)^2+4z=0$. **4.** $y^2+z^2=1$.
5. $6(x^2+y^2+z^2-1)=(x-2y-z)^2$.
8. $\phi(z+y, 2\sqrt{z-x}-y)=0$; $z=x$.
9. $\phi(z, \sqrt{z-x}-\sqrt{z-y})=0$; $z=x$, $z=y$.
10. $\phi(y+2\sqrt{x}, \sqrt{z+x}-\sqrt{x})=0$; $z=-x$.

Arts. 40–41. Page 99

1. $\phi(x_2-x_1, x_3-x_1, z-x_1)=0$.
2. $\phi(x_2+x_1, x_3-x_1, x_4+x_1, z)=0$.
3. $\phi(z/x_1, z/x_2, z/x_3)=0$.
4. $\phi(x_1z, z/x_2, x_3z)=0$.
5. $\phi(x_2^2-x_1^2, 2x_3-x_1^2, z)=0$.
6. $\phi(x_1x_2, 2\log x_2+x_3^2, \log z-x_3)=0$.
7. $\phi(x_2-x_1, x_3-x_1, z^2-2x_3)=0$.
8. $\phi(x_2/x_1, x_3/x_1, z^2-2\log x_1)=0$.
9. $\phi(x_2^2-x_1^2, x_3^2-x_1^2, z)=0$.
10. $\phi(x_2^2-x_1^2, x_3^2-x_1^2, z^2-x_1^2)=0$.
11. $\phi(x_1x_2, \log x_3-x_1x_2\log x_1, z/x_1)=0$.
12. $\phi(x_1+x_2+x_3, x_1^2+x_2^2+x_3^2, z)=0$.
13. $\phi(x_2+x_1, x_3-x_1, x_4+x_1)=0$.
14. $\phi(x_2^2-x_1^2, 2x_3-x_1^2)=0$.
15. $\phi(x_2^2-x_1^2, x_3^2-x_1^2)=0$.
16. $\phi(x_1+x_2+x_3, x_1^2+x_2^2+x_3^2)=0$.

Art. 42. Page 101

1. $\phi[(y-y_0)/(x-x_0), (z-z_0)/(x-x_0)]=0$: cones with vertices at (x_0, y_0, z_0).
2. $\phi(y/x, z)=0$: surfaces with rulings parallel to the xy-plane.
3. $\phi(y/x, z/x+\log x)=0$.
4. $\phi(z/x, z+y/z)=0$.
5. $\phi(x, y^2+z^2)=0$: surfaces of revolution with x-axis as axis.
6. $c_1(x^2+z^2)+c_2(y^2+z^2)=c_3$.
7. $x^2+y^2+z^2=c_1(x+y+z)+c_2$; centers on the line $x=y=z$.
8. $x^2+y^2+z^2=c$; centers at the origin.
9. $y^2z^4+x^2=z^2$.
10. $(x+y)^4-4(x+y)^2xyz=1$.

Art. 43. Page 104

2. $\tau=\tau_1-\dfrac{t}{\pi KL}\log\dfrac{r}{r_1}$.
3. $\tau=\tau_1-\dfrac{5(r-r_1)\cos t}{\pi KL}$.
4. $\tau=\tau_1-\dfrac{10(r^2-r_1^2)e^{-t}}{\pi KL}$.
5. $\tau=\tau_1-\dfrac{t(r-r_1)}{\pi Kr_1r}$.
6. $\tau=\tau_1-\dfrac{5(r-r_1)\sin t}{\pi K}$.
7. $\tau=\tau_1-\dfrac{10}{\pi K}e^{-2t}\log\dfrac{r}{r_1}$.
8. $u=10$, $\delta=64$.
9. $u=10+gt$, $\delta=64$.
10. $u=\dfrac{x+2gt^3+3gt^2+10}{t+1}$, $\delta=\dfrac{x-gt^3+10}{(t+1)^2}$.

Art. 45. Page 112

1. $z=\alpha y+\beta(x+\alpha)$; $z+xy=0$.
2. $z=2\alpha x+2\beta y+\alpha^2-\beta^2$; $z=y^2-x^2$.
3. $(\alpha+1)z=\alpha(\alpha+1)x-\alpha y+\beta$.
4. $(\alpha-1)z=\alpha(\alpha-1)x+(\alpha+1)y+\beta$.
5. $3z=(1+2\alpha-\alpha^2)x+3\alpha y+\beta$.
6. $z=\alpha x+(\beta-\alpha)y+\beta^2$.

7. $z = \sin\alpha \log x + \cos\alpha \log y + \beta.$

8. $16(1 - \alpha)y(z + \beta)^2 = [4\alpha x - 4(1 - \alpha)y - (z + \beta)^2]^2.$

9. $(1 + \alpha^2)\log^2 z = (x + \alpha y + \beta)^2;\ z = 0.$

10. $4\alpha(1 - z) = (x + \alpha y + \beta)^2;\quad z = 1.$

11. $4\alpha z = 3(x - y)^2 + \alpha^2(x + y) + \beta.$

12. $(1 - \alpha^2)z^4 = 4(x + \alpha y + \beta)^2.$

13. $2\alpha z + (x + \alpha y)^2 + \beta = 0.$

14. $6\alpha z = (x + y)^3 + 3\alpha^2(y - x) + \beta.$

15. $2\alpha z + (x - y)^2 - 2\alpha^2 y + \beta = 0.$

16. $z = \alpha x^2 - \alpha^2 y + \beta.$

17. $(z + \beta)^2 = y^2(4x^2 + \alpha).$

18. $2\alpha z^2 + x^2 - 4\alpha^2 y + \beta = 0.$

19. $xyz = \alpha x + \alpha^2 y + \beta xy.$

20. $z^2 = \alpha^2 x + 3\alpha y^2 + \beta.$

Arts. 46–47. Page 115

1. $z = (2 + 3\alpha - \alpha^2)x + \alpha y + \beta.$

2. $z = 2(x\cos\alpha + y\sin\alpha) + \beta.$

3. $z = 3(x\tan\alpha + y\sec\alpha) + \beta.$

4. $z = \alpha x - y\sin\alpha + \beta.$

5. $z = \alpha y - \alpha^2 x + \beta.$

6. $z = \alpha_1 y - \alpha_1^2 x + \beta_1,\quad z = \alpha_2 y - 2\alpha_2^2 x + \beta_2.$

7. $4xz = 17x^2 + 6xy + y^2.$

8. $z^2 = 4(x^2 + y^2).$

9. $z^2 = 9(y^2 - x^2).$

10. $\left(z - x \arccos \dfrac{x}{y}\right)^2 = y^2 - x^2.$

11. $4xz = y^2.$

12. $4xz = y^2;\quad 8xz = y^2.$

13. $4\alpha z = (x + \alpha y + \beta)^2;\quad z = 0.$

14. $\alpha \log^2 z = (x + \alpha y + \beta)^2;\quad z = 0.$

15. $z^2(x + \alpha y + \beta)^2 = 1 + \alpha^2;\quad z = 0.$

16. $\alpha(1 + z^2) = (x + \alpha y + \beta)^2.$

17. $\alpha^2 z^3 = 3(x + \alpha y + \beta),\ z = c.$

18. $(\log z - \alpha x - \alpha^2 y)^2 = (1 + \alpha^2)(x + \alpha y + \beta)^2;\quad z = 0.$

19. $(\alpha z^2 + x + \alpha y + \alpha x + \alpha^2 y)^2 = (1 + 6\alpha + \alpha^2)(x + \alpha y + \beta)^2.$

20. $z = \sec(x\cos\alpha + y\sin\alpha + \beta);\quad z = 0,\quad z = \pm 1.$

Arts. 48–50. Page 121

1. $z = (1 + \alpha^2)\log x + \alpha \log y + \beta.$

2. $(3z - \alpha x^3 - \beta)^2 = 4(\alpha - 1)y^3.$

3. $3\alpha z = \alpha^2 x^3 + y^3 - 3\alpha y + \beta.$

4. $(2\alpha z - y^2 - \beta)^2 = \alpha^2(\alpha + 2)x^4.$

5. $2\alpha z = (1 - \alpha)x^2 + 2\alpha^2 \log y + 8\alpha y + 2\alpha\beta.$

6. $[6\alpha(\alpha - 1)(z - y) - 3\alpha x^2 - 18\alpha^2 x - 6\beta]^2 = 16(\alpha - 1)^2(\alpha y + 1)^3.$

7. $3z + xy = 0.$

8. $24z = 3x^2 + 2y^2.$

9. $64z^3 = 27x^2y^2.$

10. $z^2 = 4x.$

11. $z^3 = 27xy.$

12. $(x^2 + y^2)z = xy.$

13. $2(\alpha - 1)z = 2\alpha(\alpha - 1)\log x + \alpha^2 y^2 + 2(\alpha - 1)\beta.$

14. $3z^3 = (\alpha^2 - 9)x + 3\alpha y + 3\beta.$

15. $\alpha \log z = \alpha^2 \log x + (1 - \alpha^2)y + \alpha\beta.$

16. $2\alpha z^2 = (\alpha^2 + 2)x^2 + 2\alpha^2 y^2 + 2\alpha\beta.$

17. $(\log z - \alpha x^2 - \beta)^2 = (1 - 4\alpha^2)\log^2 y.$

18. $2yz^2 = 2\alpha x^3 y + 4 - 9\alpha^2 + \beta y.$

19. $(e^z - \alpha e^x - \beta)^2 = (\alpha^2 - 1)e^{2y}.$

20. $\alpha \log^2 z = (\sin x + \alpha\cos y + \beta)^2.$

Art. 51. Page 126

17. $a_1 z + a_2 a_3 \log x_1 = a_1(a_1 x_2 + a_2 x_3 + a_3 x_4 + a_4).$

18. $(z - a_1 \log x_2 - a_2 \log x_3 - a_3 \log x_4 - a_4)^2 = (a_1 - a_2 + a_3)\log^2 x_1.$

19. $[a_1 x_1 + a_2 x_2 + a_3 x_3 + 1 - (a_1 + a_2)\log z]^2 = [(a_1 + a_2)^2 + a_3^2]\log^2 z.$

20. $4(a_1 x_1 + a_2 x_2 + z) = (\log x_3 + a_3)^2.$

Art. 52. Pages 129–130

1. $4z = 4a_1x_1 + 2(2 - a_1)x_2 - 3a_1^2x_2^2 + 4(a_1 - 1)\log x_3 + a_2.$

2. $2x_3z = 2a_1x_1x_3 - a_1x_2^2x_3 + a_2x_3 + a_1 + a_1^2.$

3. $z = x_1^2x_2^2 + x_1x_3 - x_2x_3 + a.$

4. $[z - x_1x_2 - \log(x_1 - x_2) + a]^2 = x_3^4.$ **5.** Inconsistent.

6. $x_2z = x_1^2x_2 + x_2x_3 + ax_2 - 4;\quad x_2z = 5x_1^2x_2 + 5x_2x_3 + bx_3 - 20.$

7. $z = a.$

8. $2x_4z = 2a_1x_1x_4 + 2a_2x_2x_4 + (a_1 - a_2)x_3^2x_4 + a_3x_4 + 2(a_1^2 + 1).$

9. $(10z - 10a_1x_1 - 5a_1^2x_2^2 - a_2)x_3^2x_4 = (a_1^2 + 3a_1 + 1)(x_2^2x_3^2x_4 + 2x_4^2 - 6x_4).$

10. $x_3x_4(z - x_1^2 + x_2 - a) = x_3 + x_4;\quad z = b.$

Art. 53. Pages 134–136

2. $12z = x^2 - 4xy + 4y^2 + 24x - 12y + 36.$

3. Planes through $(-3, 5, 2)$ parallel to the z-axis.

5. $(z - 1)^2 = 4xy.$ **6.** $x^2 + y^2 + z^2 = k^2.$ **7.** $8x(z - x) + (y - 1)^2 = 0.$

8. $z = x - 4,\ 4z = x - 1.$ **9.** $4\beta = 4b\alpha - a\alpha^2.$

10. $4\beta = 4b\alpha - a\alpha^2.$ **14.** $(x - \alpha)^2 + (y - \alpha)^2 + z^2 = \beta.$

15. $(x - \alpha^2)^2 + (y - \alpha)^2 + z^2 = \beta.$ **16.** $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1.$

17. $27xyz = 1.$ **18.** $x^{2/3} + y^{2/3} + z^{2/3} = 1.$ **19.** $x^2 + 2y^2 + 2z^2 = 2.$

Arts. 55–56. Page 145

2. $\pi = 2\sqrt{2}(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \cdots);\quad \pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots).$

3. $\pi^2 = 8\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right).$

4. $\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots);\quad \pi^2 = 8\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right);\quad f(0) = \frac{\pi}{2}.$

5. $f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n - 1)x}{2n - 1}.$

6. $f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^{n-1}\sin nx}{n} - \frac{2\cos(2n - 1)x}{(2n - 1)^2\pi}\right].$

7. $f(x) = \frac{3\pi}{8} - \frac{2}{\pi}\left(\cos x + \frac{2}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \frac{2}{6^2}\cos 6x + \cdots\right).$

8. $f(x) = -\frac{1}{\pi} + \frac{1}{2}\sin x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2 - 1}.$

9. $f(x) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n\sin nx}{4n^2 - 1}.$ **10.** $f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2 - 1}.$

Arts. 57–58. Page 150

4. $\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots)$.

5. $\pi^2 = 8\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right)$.

6. $\frac{2}{\pi}\sum_{n=1}^{\infty}\left(\cos\frac{n\pi}{2} - \cos n\pi\right)\frac{\sin nx}{n}$; $\frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n \cos (2n-1)x}{2n-1}$.

7. $\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin (4n-2)x}{2n-1}$; $\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos (2n-1)x}{2n-1}$.

8. $\left(1 + \frac{2}{\pi}\right)\sin x - \frac{1}{2}\sin 2x + \left(\frac{1}{3} - \frac{2}{3^2\pi}\right)\sin 3x - \frac{1}{4}\sin 4x +$
$\left(\frac{1}{5} + \frac{2}{5^2\pi}\right)\sin 5x - \cdots$; $\frac{3\pi}{8} - \frac{2}{\pi}\left(\cos x + \frac{2}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x\right.$
$\left. + \frac{1}{5^2}\cos 5x + \frac{2}{6^2}\cos 6x + \cdots\right)$.

9. $\frac{2}{\pi}\left[(\pi^2 - 4)\sin x - \frac{\pi^2}{2}\sin 2x + \left(\frac{\pi^2}{3} - \frac{4}{3^3}\right)\sin 3x - \frac{\pi^2}{4}\sin 4x\right.$
$\left. + \left(\frac{\pi^2}{5} - \frac{4}{5^3}\right)\sin 5x - \cdots\right]$; $\frac{\pi^2}{3} - 4\left(\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x\right.$
$\left. - \frac{1}{4^2}\cos 4x + \cdots\right)$.

10. $\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n[1 - (-1)^n e^{-\pi}]\sin nx}{1 + n^2}$; $\frac{1 - e^{-\pi}}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1 - (-1)^n e^{-\pi}]\cos nx}{1 + n^2}$.

Arts. 59–60. Page 155

4. $\frac{2}{\pi}\left[(\pi - 2)\sin x - \frac{\pi}{2}\sin 2x + \frac{\pi - 2}{3}\sin 3x - \frac{\pi}{4}\sin 4x + \cdots\right]$.

5. $\frac{\pi - 2}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right)$.

6. $-\frac{2}{\pi}\left(\sin \pi x + \frac{3}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x + \frac{3}{4}\sin 4\pi x + \cdots\right)$.

7. $1 - \frac{32}{\pi^2}\left(\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right)$.

8. $3 + \frac{8}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots\right)$.

9. $4 - \frac{6}{\pi}\left(\sin\frac{\pi x}{3} - \frac{1}{2}\sin\frac{2\pi x}{3} + \frac{1}{3}\sin\frac{3\pi x}{3} - \cdots\right)$.

10. $-\frac{32}{\pi^3}\left(\sin\frac{\pi x}{2} + \frac{\pi^2}{8}\sin\frac{2\pi x}{2} + \frac{1}{3^3}\sin\frac{3\pi x}{2} + \frac{\pi^2}{16}\sin\frac{4\pi x}{2} + \cdots\right)$.

11. $-\frac{2}{3}-\frac{16}{\pi^2}\left(\cos\frac{\pi x}{2}-\frac{1}{2^2}\cos\frac{2\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}-\cdots\right).$

12. $\frac{128}{\pi^3}\left(\sin\frac{\pi x}{4}+\frac{1}{3^3}\sin\frac{3\pi x}{4}+\frac{1}{5^3}\sin\frac{5\pi x}{4}+\cdots\right).$

13. $\frac{8}{3}-\frac{16}{\pi^2}\left(\cos\frac{\pi x}{2}+\frac{1}{2^2}\cos\frac{2\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}+\cdots\right).$

14. $\frac{2}{\pi^3}\left[(\pi^2-4)\sin\pi x+\frac{\pi^2}{2}\sin 2\pi x+\left(\frac{\pi^2}{3}-\frac{4}{3^3}\right)\sin 3\pi x+\frac{\pi^2}{4}\sin 4\pi x+\cdots\right].$

15. $\frac{1}{3}+\frac{4}{\pi^2}\left(\cos\pi x+\frac{1}{2^2}\cos 2\pi x+\frac{1}{3^2}\cos 3\pi x+\cdots\right).$

16. $2\pi\sum_{n=1}^{\infty}\frac{n[1-(-1)^n e]\sin n\pi x}{1+n^2\pi^2}.$

17. $e-1-2\sum_{n=1}^{\infty}\frac{[1-(-1)^n e]\cos n\pi x}{1+n^2\pi^2}.$

18. $2\pi\sum_{n=1}^{\infty}\frac{n[1-(-1)^n e^{-1}]\sin n\pi x}{1+n^2\pi^2}.$

19. $1-e^{-1}+2\sum_{n=1}^{\infty}\frac{[1-(-1)^n e^{-1}]\cos n\pi x}{1+n^2\pi^2}.$

20. $2\pi\sinh 1\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n\sin n\pi x}{1+n^2\pi^2};\quad 2\pi\sum_{n=1}^{\infty}\frac{n[1-(-1)^n\cosh 1]\sin n\pi x}{1+n^2\pi^2};$

$\cosh 1-1-2\sum_{n=1}^{\infty}\frac{[1-(-1)^n\cosh 1]\cos n\pi x}{1+n^2\pi^2};$

$\sinh 1+2\sinh 1\sum_{n=1}^{\infty}\frac{(-1)^n\cos n\pi x}{1+n^2\pi^2}.$

Art. 62. Pages 161–162

1. $z=f_1(3y+x)+f_2(y+4x).$
2. $z=f_1(y-x)+f_2(2y-5x).$
3. $z=f_1(y+2x)+f_2(2y-x).$
4. $z=f_1(y)+f_2(y+x)+f_3(y-x).$
5. $z=f_1(x)+f_2(y)+f_3(3y+4x).$
6. $z=f_1(y)+f_2(x)+yf_3(x)+f_4(y-2x).$
7. $z=f_1(x)+f_2(y+2x)+xf_3(y+2x).$
8. $z=f_1(y)+xf_2(y)+f_3(2y-x)+xf_4(2y-x).$
9. $z=f_1(x)+yf_2(x)+y^2f_3(x)+f_4(7y+x).$
10. $z=f_1(y)+xf_2(y)+f_3(x)+yf_4(x)+f_5(5y-6x).$
11. $z=f_1(y+2x)+xf_2(y+2x)+x^2f_3(y+2x).$
12. $z=f_1(x)+yf_2(x)+f_3(y)+f_4(y-x)+xf_5(y-x)+x^2f_6(y-x).$
13. $z=f_1(y)+e^{2y}f_2(3y-2x).$
14. $z=f_1(y+x)+e^xf_2(y+x).$
15. $z=e^{-2x}[f_1(y-x)+xf_2(y-x)].$
16. $z=xf_1(y-x)+f_2(y-x)+e^xf_3(y).$
17. $z=f_1(2y-x)+e^{3x}[xf_2(y)+f_3(y)].$
18. $z=xf_1(y)+f_2(y)+yf_3(x)+f_4(x)+e^x[xf_5(y)+f_6(y)].$

19. $z = e^x[xf_1(y) + f_2(y)] + e^y[yf_3(x) + f_4(x)]$.

20. $z = xf_1(y + x) + f_2(y + x) + e^{-x}[xf_3(y - x) + f_4(y - x)]$.

Arts. 63–64. Pages 167–168

1. $z = c_1e^{b_1y - 3b_1^2x} + \cdots + c_ne^{b_ny - 3b_n^2x}$.

2. $z = f(y) + e^y[c_1e^{a_1x + 5a_1^2y} + \cdots + c_ne^{a_nx + 5a_n^2y}]$.

3. $z = c_1e^{a_1x + a_1^2y} + \cdots + c_ne^{a_nx + a_n^2y} + C_1e^{b_1^2x + b_1y} + \cdots + C_ke^{b_k^2x + b_ky}$.

4. $z = c_1e^{a_1x - a_1^2y} + \cdots + c_ne^{a_nx - a_n^2y} + x(C_1e^{A_1x - A_1^2y} + \cdots + C_ke^{A_kx - A_k^2y})$.

5. $z = c_1e^{b_1y}\cos b_1^2x + \cdots + c_ne^{b_ny}\cos b_n^2x$
$+ C_1e^{B_1y}\sin B_1^2x + \cdots + C_ke^{B_ky}\sin B_k^2x$.

6. $z = c_1e^{a_1x}\cos a_1^2y + \cdots + c_ne^{a_nx}\cos a_n^2y + C_1e^{A_1x}\sin A_1^2y + \cdots$
$+ C_ke^{A_kx}\sin A_k^2y + x(\gamma_1e^{b_1x}\cos b_1^2y + \cdots + \gamma_Ne^{b_Nx}\cos b_N^2y$
$+ \Gamma_1e^{B_1x}\sin B_1^2y + \cdots + \Gamma_Ke^{B_Kx}\sin B_K^2y)$.

7. $1, y, y^2 - 6x, y^3 - 18xy$.

8. $e^y, xe^y, (x^2 + 10y)e^y, (x^3 + 30xy)e^y$.

9. $1, x, x^2 + 2y, x^3 + 6xy$; $y, y^2 + 2x, y^3 + 6xy$.

10. $1, x, x^2 - 2y, x^3 - 6xy$; $x^2, x^3 - 2xy, x^4 - 6x^2y$.

11. $1, y, y^2, x, y^3, xy$.

12. $1, x, x^2, y, x^3, xy$; x^4, x^2y.

13. All $x, y < 0$; $x < 0, y = 0$.

14. $x \geqq 0, y < 0$; $x < 0, y = 0$.

15. All $x, y > 0$.

16. All $x, y \geqq 0$.

Art. 65. Pages 174–175

1. $3x^2$.

2. $x^3/2 - 2x^2y$.

3. $5x^3y^2 - x^5$.

4. $2e^{2x-y}$.

5. $2xe^{x-y}$.

6. $(x^2 - 2x)e^{x+y}$.

7. $xy^2 - 9y^2/2$.

8. $4x^2y^2 + 4x^3y + 2x^4$.

9. $13x^2/4 - x^3/12$.

10. x^3y^2.

11. $3e^{y-2x}$.

12. $-8xe^{y-2x}$.

13. $2\sin(2x - 3y)$.

14. $e^{x-y}\sin(x - y)$.

15. $-e^{2x}\cos 3y$.

16. $\tan(2x + y)$.

17. $1/y$.

18. $-\log x$.

19. xy.

20. xy.

Art. 66. Pages 176–177

1. $z = f_1(xy) + f_2(y/x)$.

2. $z = f_1(xy) + f_2(xy)\log x$.

3. $z = f_1(y) + x^2f_2(xy)$.

4. $z = f_1(xy) + x^3f_2(y/x)$.

5. $z = y^2f_1(x) + yf_2(xy)$.

6. $z = f_1(x^2y) + f_2(xy^2) + xy^3$.

7. $z = f_1(y/x^3) + f_2(y^2/x) - 2(y\log x)/x^3$.

8. $z = f_1(y^3/x) + f_2(x^3y) - 2\log x\log y$.

9. $z = f_1(x^2y) + f_2(y/x) + \log^2 x\log y$.

10. $z = f_1(x^2y) + f_2(xy) + 2x^2y(\log^2 x + \log y)$.

11. $z = f_1(y/x^4) + f_2(xy) + \frac{3}{4}\sin\log y^2$.

12. $z = f_1(xy^2) + f_2(y^3/x) + \frac{7}{50}xy[7\sin\log(x/y) - \cos\log(x/y)]$.

13. $z = f_1(x) + f_2(y) + f_3(xy)$.

14. $z = f_1(y) + f_2(xy) + f_3(y/x) - 2\log x\log^2 y$.

15. $z = xf_1(y) + yf_2(x) + f_3(xy) - x^2/y$.

Art. 67. Page 182

1. $z = xf(y) + g(y) - 2\sin(x - 2y)$.
2. $z = f(x) + g(y) + x^2y - xy^3$.
3. $z = yf(x) + g(x) + xe^{-y}$.
4. $yz = xf(y) + g(y) + \log\sec x$.
5. $xz = f(x) + xg(y) - 2\log y - 3xy^2\log x$.
6. $z = yf(x) + g(x) + xy^2 - 2x^2(x + y)[\log(x + y) - 1]$.
7. $xz = f(y) + xg(y)$.
8. $z = e^{-x}f(y) + g(x) + \log y + xye^{-x}$.
9. $z = yf(x) + g(y) + 3xy^2$.
10. $z = y^2f(x) + g(x) - 2xy$.
11. $z = e^{-xy}f(x) + e^{y+xy}g(x) - 2x^2y$.
12. $z = f(y)\cos x + g(x) + \sin y$.
13. $y^2z = f(x) + g(y) + y^3\sin x$.
14. $z = e^{-y}f(x) + g(x) - \cos xy$.
15. $z = e^{xy}f(y) + g(y) + x^2$.
16. $z = f(x - y) + g(y) + x^2y^2$.
17. $z = f(x^2 + y) + g(x) + y\log x$.
18. $z = f(x^2 - y^2) + g(y) + xe^y$.
19. $xz = f(y) + x^{y+1}g(y) + x^3$.
20. $z = f(y/x) + g(x) + xy^3$.

Arts. 68–69. Pages 193–194

1. $xz = xyf(x/y) + g(y)$.
2. $z = e^yf(x - y^2) + e^{-x}g(y) + xe^y$.
3. $y(x + y)z = f(x + y) + g(y) - xy^2(x + y)$.
4. $z = f(xy) + xyg(x^2y) + x^2$.
5. $z = e^{-2xy^2}f(xy) + e^{xy^2}g(xy) - e^{xy}/2x^2y^2$.
6. $z = f(x + y) + g(xy) + 3x^2y - x^3$.
7. $z = f(x^2 + 2y) + g(2x + y^2) + x^3 + y^3 + 3xy$.
8. $z = f(x + y) + g(ye^x) + (y - 2)e^{-x}$.
9. $z = f(xe^y) + g(ye^x) - xy - x - y$.
10. $z = f(x - y) + g(xy) + 4x^2 + (x + y)^2[2\log(x + y) - 1] + 8xy^2(\log x - 1)$.
14. $yz = x^2y^2 + yf(xy) + g(x)$.
16. $xz = x^2f(x + y) + g(x + y) - x^2 - xy - x^2\log x$.

Art. 70. Pages 197–198

1. $z = 2y$.
2. $z = 3x^2 + xy + y$.
3. $z = 4x^2$.
4. $z = (x + y)^2 - 1$.
5. $z = 6x^2 + 4xy + 13y^2 - 10x - 10y + 5$.
7. $z = 3x + ay + b$.
8. $z = ax^2 + bx + cy + k$, $(x + A)z = Bx^2 + Cxy + Ex + ACy + K$.
9. $z = 2x^2 - x$.
10. $z = x^2 + 2y^2 - 3$.
11. $xz = y^2 - 4x^2$.
13. $z^2(1 - y^2 + z^2) + x^2(z^2 - y^2) = 0$.
14. $z = xy - x + y$.
15. $z = 1 + 6x - 3y - 4xy$.
16. $z = 2 - 3y + 4y^2 + 24x$.
17. $z = 3x^2 + 18xy + 27y^2 + 4x$.
18. $z = 2e^{-y}\sin x$.
19. $z = \cos(y + x) - \sin(y - x)$.

Art. 71. Pages 201–202

1. 0.26 in.
2. $y = y_0 \sin\dfrac{2\pi x}{L}\cos\dfrac{2a\pi t}{L}$ ft.
3. $y = \dfrac{Lv_0}{12a\pi}\left(9\sin\dfrac{\pi x}{L}\sin\dfrac{a\pi t}{L} - \sin\dfrac{3\pi x}{L}\sin\dfrac{3a\pi t}{L}\right)$ ft.
4. 8.1 ft./sec.
5. $y = \dfrac{y_0}{16}\left(10\sin\dfrac{\pi x}{L}\cos\dfrac{a\pi t}{L} - 5\sin\dfrac{3\pi x}{L}\cos\dfrac{3a\pi t}{L} + \sin\dfrac{5\pi x}{L}\cos\dfrac{5a\pi t}{L}\right)$ ft.

19. $z = e^x[xf_1(y) + f_2(y)] + e^y[yf_3(x) + f_4(x)]$.
20. $z = xf_1(y + x) + f_2(y + x) + e^{-x}[xf_3(y - x) + f_4(y - x)]$.

Arts. 63–64. Pages 167–168

1. $z = c_1e^{b_1y - 3b_1^2x} + \cdots + c_ne^{b_ny - 3b_n^2x}$.
2. $z = f(y) + e^y[c_1e^{a_1x + 5a_1^2y} + \cdots + c_ne^{a_nx + 5a_n^2y}]$.
3. $z = c_1e^{a_1x + a_1^2y} + \cdots + c_ne^{a_nx + a_n^2y} + C_1e^{b_1^2x + b_1y} + \cdots + C_ke^{b_k^2x + b_ky}$.
4. $z = c_1e^{a_1x - a_1^2y} + \cdots + c_ne^{a_nx - a_n^2y} + x(C_1e^{A_1x - A_1^2y} + \cdots + C_ke^{A_kx - A_k^2y})$.
5. $z = c_1e^{b_1y}\cos b_1^2x + \cdots + c_ne^{b_ny}\cos b_n^2x$
$+ C_1e^{B_1y}\sin B_1^2x + \cdots + C_ke^{B_ky}\sin B_k^2x$.
6. $z = c_1e^{a_1x}\cos a_1^2y + \cdots + c_ne^{a_nx}\cos a_n^2y + C_1e^{A_1x}\sin A_1^2y + \cdots$
$+ C_ke^{A_kx}\sin A_k^2y + x(\gamma_1e^{b_1x}\cos b_1^2y + \cdots + \gamma_Ne^{b_Nx}\cos b_N^2y$
$+ \Gamma_1e^{B_1x}\sin B_1^2y + \cdots + \Gamma_Ke^{B_Kx}\sin B_K^2y)$.

7. $1, y, y^2 - 6x, y^3 - 18xy$.
8. $e^y, xe^y, (x^2 + 10y)e^y, (x^3 + 30xy)e^y$.

9. $1, x, x^2 + 2y, x^3 + 6xy;\quad y, y^2 + 2x, y^3 + 6xy$.
10. $1, x, x^2 - 2y, x^3 - 6xy;\quad x^2, x^3 - 2xy, x^4 - 6x^2y$.

11. $1, y, y^2, x, y^3, xy$.
12. $1, x, x^2, y, x^3, xy;\quad x^4, x^2y$.
13. All $x, y < 0$; $x < 0, y = 0$.
14. $x \geqq 0, y < 0$; $x < 0, y = 0$.
15. All $x, y > 0$.
16. All $x, y \geqq 0$.

Art. 65. Pages 174–175

1. $3x^2$.
2. $x^3/2 - 2x^2y$.
3. $5x^3y^2 - x^5$.
4. $2e^{2x-y}$.
5. $2xe^{x-y}$.
6. $(x^2 - 2x)e^{x+y}$.
7. $xy^2 - 9y^2/2$.
8. $4x^2y^2 + 4x^3y + 2x^4$.
9. $13x^2/4 - x^3/12$.
10. x^3y^2.
11. $3e^{y-2x}$.
12. $-8xe^{y-2x}$.
13. $2\sin(2x - 3y)$.
14. $e^{x-y}\sin(x - y)$.
15. $-e^{2x}\cos 3y$.
16. $\tan(2x + y)$.
17. $1/y$.
18. $-\log x$.
19. xy.
20. xy.

Art. 66. Pages 176–177

1. $z = f_1(xy) + f_2(y/x)$.
2. $z = f_1(xy) + f_2(xy)\log x$.
3. $z = f_1(y) + x^2f_2(xy)$.
4. $z = f_1(xy) + x^3f_2(y/x)$.
5. $z = y^2f_1(x) + yf_2(xy)$.
6. $z = f_1(x^2y) + f_2(xy^2) + xy^3$.
7. $z = f_1(y/x^3) + f_2(y^2/x) - 2(y\log x)/x^3$.
8. $z = f_1(y^3/x) + f_2(x^3y) - 2\log x\log y$.
9. $z = f_1(x^2y) + f_2(y/x) + \log^2 x\log y$.
10. $z = f_1(x^2y) + f_2(xy) + 2x^2y(\log^2 x + \log y)$.
11. $z = f_1(y/x^4) + f_2(xy) + \frac{3}{4}\sin\log y^2$.
12. $z = f_1(xy^2) + f_2(y^3/x) + \frac{7}{50}xy[7\sin\log(x/y) - \cos\log(x/y)]$.
13. $z = f_1(x) + f_2(y) + f_3(xy)$.
14. $z = f_1(y) + f_2(xy) + f_3(y/x) - 2\log x\log^2 y$.
15. $z = xf_1(y) + yf_2(x) + f_3(xy) - x^2/y$.

Art. 67. Page 182

1. $z = xf(y) + g(y) - 2 \sin (x - 2y)$.
2. $z = f(x) + g(y) + x^2y - xy^3$.
3. $z = yf(x) + g(x) + xe^{-y}$.
4. $yz = xf(y) + g(y) + \log \sec x$.
5. $xz = f(x) + xg(y) - 2 \log y - 3xy^2 \log x$.
6. $z = yf(x) + g(x) + xy^2 - 2x^2(x + y)[\log (x + y) - 1]$.
7. $xz = f(y) + xg(y)$.
8. $z = e^{-x}f(y) + g(x) + \log y + xye^{-x}$.
9. $z = yf(x) + g(y) + 3xy^2$.
10. $z = y^2f(x) + g(x) - 2xy$.
11. $z = e^{-xy}f(x) + e^{y + xy}g(x) - 2x^2y$.
12. $z = f(y) \cos x + g(x) + \sin y$.
13. $y^2z = f(x) + g(y) + y^3 \sin x$.
14. $z = e^{-y}f(x) + g(x) - \cos xy$.
15. $z = e^{xy}f(y) + g(y) + x^2$.
16. $z = f(x - y) + g(y) + x^2y^2$.
17. $z = f(x^2 + y) + g(x) + y \log x$.
18. $z = f(x^2 - y^2) + g(y) + xe^y$.
19. $xz = f(y) + x^{y+1}g(y) + x^3$.
20. $z = f(y/x) + g(x) + xy^3$.

Arts. 68–69. Pages 193–194

1. $xz = xyf(x/y) + g(y)$.
2. $z = e^yf(x - y^2) + e^{-x}g(y) + xe^y$.
3. $y(x + y)z = f(x + y) + g(y) - xy^2(x + y)$.
4. $z = f(xy) + xyg(x^2y) + x^2$.
5. $z = e^{-2xy^2}f(xy) + e^{xy^2}g(xy) - e^{xy}/2x^2y^2$.
6. $z = f(x + y) + g(xy) + 3x^2y - x^3$.
7. $z = f(x^2 + 2y) + g(2x + y^2) + x^3 + y^3 + 3xy$.
8. $z = f(x + y) + g(ye^x) + (y - 2)e^{-x}$.
9. $z = f(xe^y) + g(ye^x) - xy - x - y$.
10. $z = f(x - y) + g(xy) + 4x^2 + (x + y)^2 [2 \log (x + y) - 1] + 8xy (\log x - 1)$.
14. $yz = x^2y^2 + yf(xy) + g(x)$.
16. $xz = x^2f(x + y) + g(x + y) - x^2 - xy - x^2 \log x$.

Art. 70. Pages 197–198

1. $z = 2y$.
2. $z = 3x^2 + xy + y$.
3. $z = 4x^2$.
4. $z = (x + y)^2 - 1$.
5. $z = 6x^2 + 4xy + 13y^2 - 10x - 10y + 5$.
7. $z = 3x + ay + b$.
8. $z = ax^2 + bx + cy + k$, $(x + A) z = Bx^2 + Cxy + Ex + ACy + K$.
9. $z = 2x^2 - x$.
10. $z = x^2 + 2y^2 - 3$.
11. $xz = y^2 - 4x^2$.
13. $z^2(1 - y^2 + z^2) + x^2(z^2 - y^2) = 0$.
14. $z = xy - x + y$.
15. $z = 1 + 6x - 3y - 4xy$.
16. $z = 2 - 3y + 4y^2 + 24x$.
17. $z = 3x^2 + 18xy + 27y^2 + 4x$.
18. $z = 2e^{-y} \sin x$.
19. $z = \cos (y + x) - \sin (y - x)$.

Art. 71. Pages 201–202

1. 0.26 in.
2. $y = y_0 \sin \dfrac{2\pi x}{L} \cos \dfrac{2a\pi t}{L}$ ft.
3. $y = \dfrac{Lv_0}{12a\pi}\left(9 \sin \dfrac{\pi x}{L} \sin \dfrac{a\pi t}{L} - \sin \dfrac{3\pi x}{L} \sin \dfrac{3a\pi t}{L}\right)$ ft.
4. 8.1 ft./sec.
5. $y = \dfrac{y_0}{16}\left(10 \sin \dfrac{\pi x}{L} \cos \dfrac{a\pi t}{L} - 5 \sin \dfrac{3\pi x}{L} \cos \dfrac{3a\pi t}{L} + \sin \dfrac{5\pi x}{L} \cos \dfrac{5a\pi t}{L}\right)$ ft.

6. 0.50 in. **7.** $y = \dfrac{8KL^3}{a\pi^4}\left(\sin\dfrac{\pi x}{L}\sin\dfrac{a\pi t}{L} + \dfrac{1}{3^4}\sin\dfrac{3\pi x}{L}\sin\dfrac{3\pi at}{L} + \cdots\right)$ ft.

8. 0.19 in. **9.** $y = \dfrac{4cL^2}{a\pi^3}\left(\sin\dfrac{\pi x}{L}\sin\dfrac{a\pi t}{L} - \dfrac{1}{3^3}\sin\dfrac{3\pi x}{L}\sin\dfrac{3\pi at}{L} + \cdots\right)$ ft.

10. 0.18 in.

Art. 72. Page 206

2. 39.7°. **3.** $\tau = \dfrac{10}{\pi}\left(20e^{-\pi^2\alpha^2t/900}\sin\dfrac{\pi x}{30} - 6e^{-4\pi^2\alpha^2t/900}\sin\dfrac{2\pi x}{30} + \dfrac{20}{3}e^{-9\pi^2\alpha^2t/900}\sin\dfrac{3\pi x}{30} - \dfrac{6}{2}e^{-16\pi^2\alpha^2t/900}\sin\dfrac{4\pi x}{30} + \cdots\right)$ deg.

4. 34.1°. **5.** $\tau = 50 + \dfrac{5}{4}x - \dfrac{50}{\pi}\left(6e^{-\pi^2\alpha^2t/1600}\sin\dfrac{\pi x}{40} - e^{-4\pi^2\alpha^2t/1600}\sin\dfrac{2\pi x}{40} + \dfrac{6}{3}e^{-9\pi^2\alpha^2t/1600}\sin\dfrac{3\pi x}{40} - \dfrac{1}{2}e^{-16\pi^2\alpha^2t/1600}\sin\dfrac{4\pi x}{40} + \cdots\right)$ deg.

6. 11.0°. **7.** 6.63 min.

8. $\tau = 50 - \dfrac{400}{\pi^2}\left(e^{-\pi^2\alpha^2t/2500}\cos\dfrac{\pi x}{50} + \dfrac{1}{3^2}e^{-9\pi^2\alpha^2t/2500}\cos\dfrac{3\pi x}{50} + \dfrac{1}{5^2}e^{-25\pi^2\alpha^2t/2500}\cos\dfrac{5\pi x}{50} + \cdots\right)$ deg.

10. 1.40 hr.

Art. 73. Pages 209–210

1. 5.9 cm.

2. $\tau = \dfrac{3200}{\pi^3}\left[\left(\dfrac{\rho}{10}\right)\sin\theta + \dfrac{1}{3^3}\left(\dfrac{\rho}{10}\right)^3\sin 3\theta + \cdots\right]$ deg.

3. 5.1 cm.

4. $\tau = \dfrac{3200}{\pi^3}\left[e^{-\pi y/10}\sin\dfrac{\pi x}{10} + \dfrac{1}{3^3}e^{-3\pi y/10}\sin\dfrac{3\pi x}{10} + \cdots\right]$ deg.

5. 21.4°; 22.1 cm.

6. $\tau = \dfrac{800}{\pi^2}\left(e^{-\pi y/10}\sin\dfrac{\pi x}{10} - \dfrac{1}{3^2}e^{-3\pi y/10}\sin\dfrac{3\pi x}{10} + \cdots\right)$ deg.

7. 25.4°.

8. $\tau = \dfrac{3200}{\pi^3}\left(\dfrac{\sin\dfrac{\pi x}{20}\sinh\dfrac{\pi y}{20}}{\sinh\pi} + \dfrac{\sin\dfrac{3\pi x}{20}\sinh\dfrac{3\pi y}{20}}{3^3\sinh 3\pi} + \cdots\right)$ deg.

9. 20.5°.

Art. 74. Pages 213–214

2. 0.061 amp. **4.** $E = 1100 - 11x + \dfrac{2000}{\pi}\left(e^{-10\pi^2t}\sin\dfrac{\pi x}{100} - \dfrac{1}{2}e^{-40\pi^2t}\sin\dfrac{2\pi x}{100} + \dfrac{1}{3}e^{-90\pi^2t}\sin\dfrac{3\pi x}{100} - \cdots\right)$ volts.

5. 0.39 amp.; 1.88 amp.

6. $E = 1000 - 10x - \frac{2000}{\pi}\left(e^{-10\pi^2 t}\sin\frac{\pi x}{100} + \frac{1}{2}e^{-40\pi^2 t}\sin\frac{2\pi x}{100} + \frac{1}{3}e^{-90\pi^2 t}\sin\frac{3\pi x}{100} + \cdots\right)$ volts.

7. 29.3%, 72.3%, 98.6%.

8. $E = 100 - 10x + \frac{200}{\pi}\left(\sin\frac{\pi x}{10}\cos 100\pi t - \frac{1}{2}\sin\frac{2\pi x}{10}\cos 200\pi t + \frac{1}{3}\sin\frac{3\pi x}{10}\cos 300\pi t + \cdots\right)$ volts.

9. $I = 10t - \frac{1}{5\pi}\left(\cos\frac{\pi x}{10}\sin 100\pi t - \frac{1}{2}\cos\frac{2\pi x}{10}\sin 200\pi t + \cdots\right)$ amp.; 0.25 amp.

Art. 76. Page 222

1. $p = \phi(z)$.
2. $q - p = \phi_1(z)$; $q = p\phi_2(x + y)$.
3. $p = q\phi(z)$.
4. $xp + xy^2 = \phi(xy)$.
5. $p = 2xy^2 + \phi(y - x)$.
6. $q = \log x + \phi(x^2 + y)$.
7. $p = e^y + x\phi(x^2 - y^2)$.
8. $yq = 3xy^3 + \phi(y/x)$.
9. $p - q = \phi_1(x + z)$; $p + 1 = q\phi_2(x + y)$.
10. $xp + z = x\phi(x/y)$.
11. $p + z = y^2e^y + e^y\phi(x - y^2)$.
12. $1 + p + q = x\phi_1(y + z)$; $x(1 + q) = (1 + p + q)\phi_2(x - y)$.
13. $q = xp\phi_1(xz)$; $xp + z = q\phi_2(z)$.
14. $(q - xp)e^y = q\phi(z)$.
15. $q - 1 = p\phi_1(x + z)$; $p + 1 = q\phi_2(y - z)$.

Art. 77. Page 228

1. $p + 3y = \phi_1(2q + x)$; $2p + y = \phi_2(q + 3x)$.
2. $p + 2y = \phi_1(q - x)$; $p - y = \phi_2(q + 2x)$.
3. $p - 2x + y = \phi_1(q - x - y)$; $p - 2x - y = \phi_2(q + x - y)$.
4. $p + x + y = \phi(q + x + y)$.
5. $p - 2x + y = \phi(q + x - 2y)$.
6. $p + x = \phi_1(q - x - 2y)$; $p + x - y = \phi_2(q - 2y)$.
7. $p - 2x = \phi(q + y)$.
8. $p + y^2 = \phi(q - x^2)$.
9. $yp + x = \phi(q + y)$.
10. $xp - q = \phi(yq + p)$.

Art. 78. Pages 231–232

1. $x = f(z) + g(y)$.
2. $y = f(z) + g(x + y)$.
3. $y + xf(z) = g(z)$.
4. $z + xy^2 = f(xy) + g(y)$.
5. $z = x^2y^2 + f(y - x) + g(y)$.
6. $z = y\log x + f(x^2 + y) + g(x)$.
7. $z = xe^y + f(x^2 - y^2) + g(y)$.
8. $z = xy^3 + f(y/x) + g(x)$.
9. $z = f(x + z) + g(x + y)$.
10. $xz = y^2f(x/y) + g(y)$.
11. $z = y^2e^y + e^yf(x - y^2) + e^{-x}g(y)$.
12. $x^2 = f(y + z) + g(x - y)$.
13. $y = f(xz) + g(z)$.
14. $xe^y = xf(z) + g(z)$.
15. $x = f(x + z) + g(y - z)$.
16. $5x = 2Q - P$, $5y = 2f'(P) - g'(Q)$, $10z = -30xy + 10yQ - 2f(P) + 2g(Q)$.
17. $3x = Q - P$, $3y = f'(P) - g'(Q)$, $3z = 3xy + 3yP - f(P) + g(Q)$.

18. $2x = Q - P, 2y = f'(P) - g'(Q), 2z = 2x^2 - 2xy + y^2 + 2yQ - f(P) + g(Q)$.
19. $z = (a^2 - 1)x^2/2 + (a - 1)xy + bx + f(y + ax)$.
20. $z = (1 - a^2)x^2 - (2a + 1)xy + bx + f(y + ax)$.
21. $x = Q - P, y = f'(P) - g'(Q), z = y^2 - x^2/2 + yQ - f(P) + g(Q)$.
22. $z = (a^2 + 2)x^2/2 + axy + bx + f(y + ax)$.
23. $z = - a(a + 1)x^3/3 - ax^2y - xy^2 + bx + f(y + ax)$.
24. $6a^2z = 2y^3 - 3a^2y^2 + 6axy + by + f(y^2 + 2ax)$.
25. $z = b \log (x - a) + f[(x - a)^a(1 + ay)]$.

Art. 79. Page 234

1. $z = ay + f(x + 2y), z = ay + g(x - 2y)$.
2. $z = ay + f(x + y), z = ay + g(x + 2y)$.
3. $z = ay + f(3x + y), z = ay + g(3x - y)$.
4. $z = ay + f(x + y)$.
5. $z = ay + f(2x + y)$.
6. $z = ay + f_1(x + y), z = ay + f_2(x - y), z = ay + f_3(x + iy), z = ay + f_4(x - iy)$.
7. $z = ax + by + c$.
8. $z = ax + by + c$.
9. $z = f(x + ay)$.
10. $z = ay + f_1(x), \quad z = ay + f_2(x + y), \quad z = ay + f_3(x - y)$,
$z = ay + f_4(x + 2y), z = ay + f_5(x - 2y)$.

Art. 80. Page 237

1. $z = y^2/2 + ay + f(x)$.
2. $z = ax + f(y)$.
3. $z = f(y)$.
4. $z = f(x), z = ax + g(y)$.
5. $z = ax + f(y)$.
6. $z = ay - y^2 + f(x)$.
7. $z = A + Bx + Cy + (1 - E + E^2)x^2/2E + Exy + Fy^2$.
8. $z = ay + f(x); \quad z = bx + g(y)$.
9. $z = A + Bx + Cy + (E^2 - 2F)x^2/2 + Exy + Fy^2$.
10. $z = ax + f(x - y)$.

Art. 81. Pages 238–239

6. $(y + z)^2 = x + 1$.
7. $2y^2 + z^2 - 2xy + 2yz = 0$.
8. $(2z + y)^2 = 4(x - 2y); \quad (z + 2y)^2 = x - 2y$.
9. $z = 2x + y^2$.
10. $(x - 2y)^2 + 4z^2 = 4, (x - 2y)^2 + 4(z - x)^2 = 4$.

INDEX